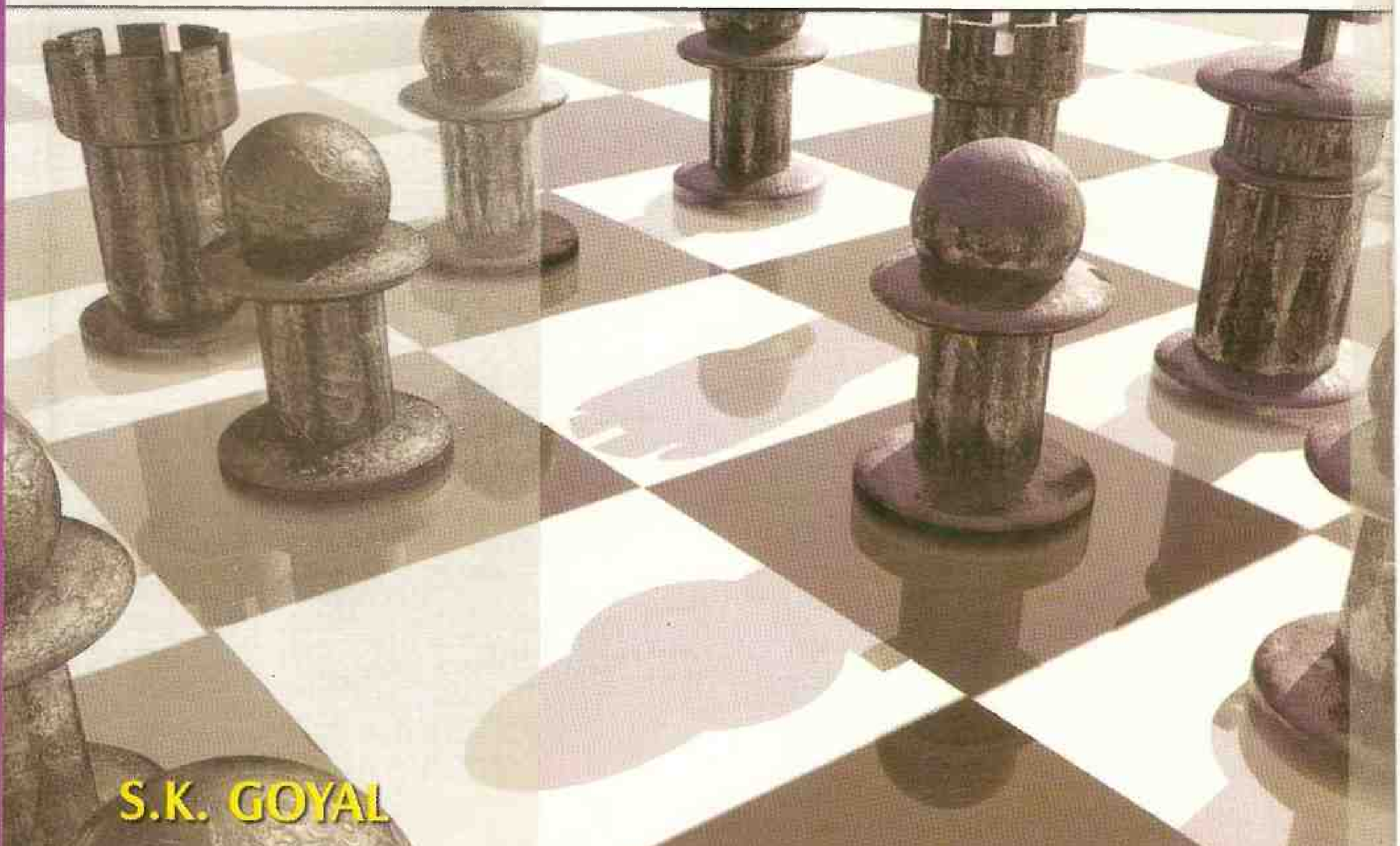
OBJECTIVE

MATHEMATICS



S.K. GOYAL

TARGET IIT 2006-07

IIT SCREENING

Arihant OBJECTIVE MATHEMATICS IIT SCREENING S.K. GOYAL

Arihant

OBJECTIVE

MATHEMATICS

IIT SCREENING



S.K. GOYAL

Director

Omega Classes, Meerut

ARIHANT PRAKASHAN

KALINDI, T.P. NAGAR, MEERUT-250 002

TARGET IIT 2006-07

MATHEMATICS GALAXY

- *A Text Book of* Algebra — S.K. Goyal
- *A Text Book of* Co-ordinate Geometry — S.K. Goyal
- *Interactive* Algebra (Vol. I & II) — S.K. Goyal
- *A Text Book of* Calculus (Differential) — Amit M. Agarwal
- *A Text Book of* Calculus (Integral) — Amit M. Agarwal
- *A Text Book of* Vector & 3D Geometry — Amit M. Agarwal
- Play with Graphs — Amit M. Agarwal
- *A Text Book of* Trigonometry — Amit M. Agarwal
- Problems in Mathematics — S.K. Goyal
- *Objective* Mathematics — S.K. Goyal

ARIHANT PRAKASHAN
An ISO 9001:2000 Organisation
Kalindi, Transport Nagar
Baghpat Road, MEERUT-250 002 (U.P.)
Tel. : (0121) 2401479, 2512970, 2402029
Fax : (0121) 2401648
email : info@arihantbooks.com
on web : www.arihantbooks.com

© Author

All the rights reserved. No part of this publication may be reproduced, stored in a retrieval system transmitted in any form or by any means- electronic, mechanical, photocopying recording or otherwise, without prior permission of the author and publisher.

ISBN 818348014-4

Price : Rs. 200.00

Laser Typesetting at:
Vibgyor Computer

Printed at:
Sanjay Printers

PREFACE

This new venture is intended for recently introduced Screening Test in new system of Entrance Examination of IIT-JEE. This is the first book of its kind for this new set up. It is in continuation of my earlier book "Problems in Mathematics" catering to the needs of students for the main examination of IIT-JEE.

- Major changes have been effected in the set up the book in the edition.
- The book has been divided into 33 chapters.
- In each chapter, first of all the theory in brief but having all the basic concepts/formulae is given to make the student refresh his memory and also for clear understanding.
- Each chapter has both set of multiple choice questions-having one correct alternative, and one or more than one correct alternatives.
- At the end of each chapter a practice test is provided for the student to assess his relative ability on the chapter.
- Hints & Solutions of selected questions have been provided in the end of book.
- The number of questions has been increased two fold and now there are more than 2000 questions.

I am extremely thankful to Shri Yogesh Chand Jain of M/s Arihant Prakashan, Meerut for their all out efforts to bring out this book in best possible form. I also place, on record my thank to Shri Raj Kumar (for designing) and M/s Vibgyor Computers (for laser typesetting).

Suggestions for the improvement of the book are, of course, cordially invited.

- S.K. Goyal

CONTENTS

ALGEBRA

1

COMPLEX NUMBERS

§ 1.1. Important Points

1. Property of Order : *i.e.* $(a+ib) <$ (or $>$) $c+id$ is not defined. For example, the statement $9+6i < 2-i$ makes no sense.

Note : (i) Complex numbers with imaginary parts zero are said to be **purely real** and similarly those with real parts zero are said to be **purely imaginary.**

(ii) **iota :** $\because i = \sqrt{-1}$ is called the imaginary unit.

Also $i^2 = -1, i^3 = -i, i^4 = 1$, etc.

In general $i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1$

$i^{4n+3} = -i$ for any integer n

For example, $i^{1997} = i^{4\times 499+1} = i$

Also, $i = \frac{-1}{i}$,

2. A complex number z is said to be *purely real* if $I_m(z) = 0$ and is said to be purely imaginary if Re $(z) = 0$. The complex number $0 = 0 + i.\,0$ is both purely real and purely imaginary.
3. The sum of four consecutive powers of i is zero.

Ex. $\sum_{n=1}^{4n+7} i^n = i + i^2 + i^3 + \sum_{n=4}^{4n+7} i^n = i - 1 - i + 0 = -1$

4. To find digit in the unit's place, this method is clear from following example :

Ex. What is the digit in the unit's place of $(143)^{86}$?

Sol. The digits in unit's place of different powers of 3 are as follows :

3, 9, 7, 1, 3, 9, 7, 1, ... (period being 4)

remainder in $86 \div 4 = 2$

So the digit in the unit's place of $(143)^{86} = 9$

[Second term in the sequence of 3, 9, 7, 1, ...]

5. $\sqrt{-a} = i\sqrt{a}$, when 'a' is any real number. Keeping this result in mind the following computation is correct.

$\sqrt{-a}\,\sqrt{-b} - i\sqrt{a}\,i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}$

But th the computation $\sqrt{-a}\,\sqrt{-b} = \sqrt{\{(-a)(-b)\}} = \sqrt{ab}$ is wrong.

Because the property $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ hold good only if, at least one of $\sqrt{a}$ or $\sqrt{b}$ is real. It does not hold good if a, b are negative numbers, *i.e.*, $\sqrt{a}$, $\sqrt{b}$ are imaginary numbers.

§ 1.2. Conjugate Complex Number

The complex number $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where a and b are real numbers, $i = \sqrt{-1}$ and $b \neq 0$ are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of i).

Properties of Conjugate : z is the mirror image of z along real axis.

(i) $(\overline{\bar{z}}) = z$

(ii) $z = \bar{z} \Leftrightarrow z$ is purely real

(iii) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary

(iv) $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$

(v) $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$

(vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(viii) $\overline{z_1 z_2} = \bar{z}_1 \,.\, \bar{z}_2$

(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

(x) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\text{Re}(z_1\bar{z}_2) = 2\text{Re}(\bar{z}_1 z_2)$

(xi) $z' = (z)^n$

(xii) If $z = f(z_1)$ then $\bar{z} = f(\bar{z}_1)$

§ 1.3. Principal Value of Arg z

If $z = a + ib$, $a, b \in R$, then arg $z = \tan^{-1}(b/a)$ always gives the principal value. It depends on the quadrant in which the point (a, b) lies :

(i) $(a, b) \in$ first quadrant $a > 0, b > 0$, the principal value $= \arg z = \theta = \tan^{-1}\left|\dfrac{b}{a}\right|$

(ii) $(a, b) \in$ second quadrant $a < 0, b > 0$, the principal value

$$= \arg z = \theta = \pi - \tan^{-1}\left|\frac{b}{a}\right|$$

(iii) $(a, b) \in$ third quadrant $a < 0, b < 0$, the principal value

$$= \arg z = \theta = -\pi + \tan^{-1}\left|\frac{b}{a}\right|$$

(iv) $(a, b) \in$ fourth quadrant $a > 0, b < 0$, the principal value

$$= \arg z = \theta = -\tan^{-1}\left|\frac{b}{a}\right|$$

Note.

(i) $-\pi < \theta < \pi$

(ii) amplitude of the complex number 0 is not defined

(iii) If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and amp z_1 = amp z_2 .

(iv) If arg $z = \pi/2$ or $-\pi/2$, is purely imaginary; if arg $z = 0$ or $\pm\pi$, z is purely real.

§ 1.4 Coni Method

If z_1, z_2, z_3 be the affixes of the vertices of a triangle ABC described in counter-clockwise sense (Fig. 1.1) then :

$$\frac{(z_1 - z_2)}{|z_1 - z_2|} e^{i\alpha} = \frac{(z_1 - z_3)}{|z_1 - z_3|}$$

or
$$\text{amp}\left(\frac{z_1 - z_3}{z_1 - z_2}\right) = \alpha = \angle BAC$$

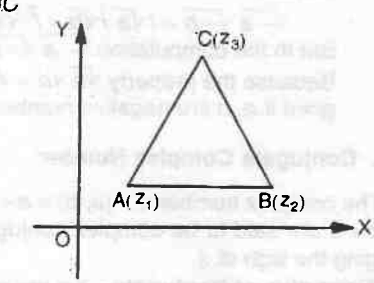


Fig. 1.1.

Note that if $\alpha = \dfrac{\pi}{2}$ or $-\dfrac{\pi}{2}$ then

$\frac{z_1 - z_3}{z_1 - z_2}$ is purely imaginary.

Note : Here only principal values of arguments are considered.

§ 1.5. Properties of Modulus

(i) $|z| > 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.

(ii) $-|z| < \text{Re}(z) < |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

In general $|z_1 z_2 z_3 z_4 \ldots z_n| = |z_1||z_2||z_3| \ldots |z_n|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(vii) $|z_1 + z_2| < |z_1| + |z_2|$

In general $|z_1 + z_2 + z_3 \pm \ldots + z_n| < |z_1| + |z_2| + |z_3| + \ldots + |z_n|$

(vi) $|z_1 - z_2| > ||z_1| - |z_2||$

(ix) $|z^n| = |z|^n$

(x) $||z_1| - |z_2|| < |z_1 + z_2| < |z_1| + |z_2|$

Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1| + |z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$.

(xi) $|z_1 + z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta = \arg(z_2)$.

(xiii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xiv) **Unimodular :** *i.e.*, unit modulus

If z is unimodular then $|z| = 1$. In case of unimodular let $z = \cos\theta + i\sin\theta$, $\theta \in R$.

Note : $\frac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$.

§ 1.6. Properties of Argument

(i) $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

In general $\text{Arg}(z_1 z_2 z_3 \ldots z_n) = \text{Arg}(z_1) + Arg(z_2) + Arg(z_3) + \ldots + \text{Arg}(z_n)$

(ii) $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$

(iii) $\text{Arg}\left(\frac{z}{\bar{z}}\right) = 2\,\text{Arg}\,z$

(iv) $\text{Arg}(z^n) = n\,\text{Arg}(z)$

(v) If $\text{Arg}\left(\frac{z_2}{z_1}\right) = \theta$, then $\text{Arg}\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in I$.

(vi) $\text{Arg}\,\bar{z} = -\text{Arg}\,z$

§ 1.7. Problem Involving the nth Root of Unity

Unity has n roots *viz.* $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$ which are in G.P., and about which we find. The sum of these n roots is zero.

(a) Here $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$ is the basic concept to be understood.

(b) The product of these n roots is $(-1)^{n-1}$.

§ 1.8. Demoivre's Theorem

(a) If n is a positive or negative integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(b) If n is a positive integer, then

$$(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

where $k = 0, 1, 2, 3, \ldots, (n-1)$

The value corresponding to $k = 0$ is called the principal value.

Demoivre's theorem is valid if n is any rational number.

§1.9. Square Root of a Complex Number

The square roots of $z = a + ib$ are :

$$\pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b > 0$$

and $$\pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|+a}{2}}\right] \text{ for } b < 0.$$

Notes :

1. The square root of i are : $\pm\left(\frac{1+i}{\sqrt{2}}\right)$. (Here $b = 1$).
2. The square root of $-i$ are : $\pm\left(\frac{1-i}{\sqrt{2}}\right)$. (Here $b = -1$).
3. The square root of ω are : $\pm\omega^2$
4. The square root of ω^2 are : $\pm\omega$

§ 1.10. Cube roots of Unity

Cube roots of unity are $1\omega, \omega^2$

Properties :

(1) $1 + \omega + \omega^2 = 0$

(2) $\omega^2 = 1$

(3) $\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2$; **ex.** $\omega^{1999} = \omega^{666\times3+1} = \omega$.

(4) $\bar{\omega} = \omega^2$ and $(\omega)^2 = \omega$

(5) A complex number $a + ib$, for which $|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$, can always be expressed in terms of i, ω or ω^2.

(6) The cube roots of unity when represented on complex plane lie on vertices of an equilateral inscribed in a unit circle, having centre origin. One vertex being on positive real axis.

(7) $a + b\omega + c\omega^2 = 0 \Rightarrow a = b = b = c$ if a, b, c are real.

(8) $\omega\bar{\omega} = \omega^3$, $\omega = e^{2\pi i/3}$, $\omega = e^{-2\pi i/3}$

§ 1.11. Some Important Results

(i) If z_1 and z_2 are two complex numbers, then the distance between z_1 and z_2 is $|z_1 - z_2|$.

(ii) Segment joining points $A(z_1)$ and $B(z_2)$ is divided by point $P(z)$ in the ratio $m_1 : m_2$

then $$z = \frac{m_1z_2 + m_2z_1}{(m_1+m_2)}, \; m_1 \text{ and } m_2 \text{ are real.}$$

(iii) The equation of the line joining z_1 and z_2 is given by

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \text{ (non parametric form)}$$

(iv) Three points z_1, z_2 and z_3 are collinear if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

(v) $\bar{a}z + a\bar{z}$ = real describes equation of a straight line.

Note : The complex and real slopes of the line $\bar{a}z + a\bar{z} + b = 0$ are $(b \in R)$

$$-\frac{a}{\bar{a}} \text{ and } -\frac{\text{Re}(a)}{\text{Im}(\bar{a})} \text{ are respectively.}$$

(a) If α_1 and α_2 are complex slopes of two lines on the Argand plane then

* If lines are perpendicular then $\alpha_1 + \alpha_2 = 0$

*If lines are parallel then $\alpha_1 = \alpha_2$

(vi) $|z - z_0| = r$ is equation of a circle, whose centre is z_0 and radius is r and $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represents the exterior of the circle $|z - z_0| = r$.

(vii) $z\bar{z} + a\bar{z} + \bar{a}z + k = 0$; (k is real) represent circle with centre $-a$ and radius $\sqrt{|a|^2 - k}$.

(viii) $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$ is equation of circle with diameter AB where $A(z_1)$ and $B(z_2)$.

(ix) If $|z - z_1| + |z - z_2| = 2a$ where $2a > |z_1 - z_2|$ then point z describes an ellipse having foci at z_1 and z_2 and $a \in R^+$.

(x) If $|z - z_1| - |z - z_2| = 2a$ where $2a < |z_1 - z_2|$ then point z describes a hyperbola having foci at z_1 and z_2 and $a \in R^+$.

(xi) Equation of all circle which are orthogonal to $|z - z_1| = r_1$ and $|z - z_2| = r_2$.

Let the circle be $|z - \alpha| = r$ cut given circles orthogonally

$$\Rightarrow \quad r^2 + r_1^2 = |\alpha - z_1|^2 \quad \text{...(1)}$$

and

$$r^2 + r_2^2 = |\alpha - z_2|^2 \quad \text{...(2)}$$

On solving $r_2^2 - r_1^2 = \alpha(\bar{z}_1 - \bar{z}_2) + \bar{\alpha}(z_1 - z_2) + |z_2|^2 - |z_1|^2$

and let $\alpha = a + ib$.

(xii) $\left|\frac{z - z_1}{z - z_2}\right| = k$ is a circle if $k \neq 1$, and is a line if $k = 1$.

(xiii) The equation $|z - z_1|^2 + |z - z_2|^2 = k$, will represent a circle if $k > \frac{1}{2}|z_1 - z_2|^2$

(xiv) If $\text{Arg}\left(\frac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\right) = \pm\pi, 0$, then points z_1, z_2, z_3, z_4 are concyclic.

§ 1.12. Important Results to Remember

(i) Iota (i) is neither 0, nore greater than 0, nor less than 0.

(ii) Amp z − Amp $(-z) = \pm\pi$ according as Amp (z) is positive or negative.

(iii) The triangle whose vertices are z_1, z_2, z_3 is equilateral iff

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

or

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

(iv) If $\left|z + \frac{1}{z}\right| = a$, the greatest and least value of $|z|$ are respectively $\frac{a + \sqrt{(a^2 + 4)}}{2}$ and $\frac{-a + \sqrt{(a^2 + 4)}}{2}$.

MULTIPLE CHOICE–I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. One of the values of i^i is $(i = \sqrt{-1})$
   (a) $e^{-\pi/2}$ (b) $e^{\pi/2}$
   (c) e^{π} (d) $e^{-\pi}$

2. If $x_r = \cos(\pi/3^r) - i\sin(\pi/3^r)$, then value of $x_1 . x_2 . \ldots \infty$, is
   (a) 1 (b) –1
   (c) $-i$ (d) i

3. The area of the triangle on the Argand plane formed by the complex numbers $-z$, iz, $z - iz$, is
   (a) $\frac{1}{2}|z|^2$ (b) $|z|^2$
   (c) $\frac{3}{2}|z|^2$ (d) None of these

4. Let $z_1 = 6 + i$ and $z_2 = 4 - 3i$. Let z be a complex number such that
   $$\arg\left(\frac{z - z_1}{z_2 - z}\right) = \frac{\pi}{2}; \text{ then } z \text{ satisfies}$$
   (a) $|z - (5 - i)| = 5$ (b) $|z - (5 - i)| = \sqrt{5}$
   (c) $|z - (5 + i)| = 5$ (d) $|z - (5 + i)| = \sqrt{5}$

5. The number of solutions of the equation
   $$z^2 + |z|^2 = 0, \quad \text{where } z \in C \text{ is}$$
   (a) one (b) two
   (c) three (d) infinitely many

6. If $z = (\lambda + 3) + i\sqrt{(5 - \lambda^2)}$; then the locus of z is
   (a) a straight line (b) a circle
   (c) an ellipse (d) a parabola

7. The locus of z which sotisfies the inequality $\log_{0.3}|z - 1| > \log_{0.3}|z - i|$ is given by
   (a) $x + y < 0$ (b) $x + y > 0$
   (c) $x - y > 0$ (d) $x - y < 0$

8. If z_1 and z_2 are any two complex numbers, then
   $$|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_1 - \sqrt{z_1^2 - z_2^2}| \text{ is equal to}$$
   (a) $|z_1|$ (b) $|z_2|$
   (c) $|z_1 + z_2|$ (d) None of these

9. If z_1 and z_2 are complex numbers satisfying
   $$\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1 \text{ and } \arg\left(\frac{z_1 - z_2}{z_1 + z_2}\right) \neq m\pi \; (m \in I)$$
   then z_1/z_2 is always
   (a) zero
   (b) a rational number
   (c) a positive real number
   (d) a purely imaginary number

10. If $z \neq 0$, then $\int_{x=0}^{100} [\arg|z|]\, dx$ is :
    (where [.] denotes the greatest integer function)
    (a) 0 (b) 10
    (c) 100 (d) not defined

11. The centre of square $ABCD$ is at $z = 0$. A is z_1. Then the centroid of triangle ABC is
    (a) $z_1(\cos\pi \pm i\sin\pi)$
    (b) $\frac{z_1}{3}(\cos\pi \pm i\sin\pi)$
    (c) $z_1(\cos\pi/2 \pm i\sin\pi/2)$
    (d) $\frac{z_1}{3}(\cos\pi/2 \pm i\sin\pi/2)$

12. The point of intersection of the curves $\arg(z - 3i) = 3\pi/4$ and $\arg(2z + 1 - 2i) = \pi/4$ is
    (a) $1/4\,(3 + 9i)$ (b) $1/4\,(3 - 9i)$
    (c) $1/2\,(3 + 2i)$ (d) no point

13. If $S(n) = i^n + i^{-n}$, where $i = \sqrt{-1}$ and n is an integer, then the total number of distinct values of $S(n)$ is
    (a) 1 (b) 2
    (c) 3 (d) 4

14. The smallest positive integer n for which $\left(\frac{1 + i}{1 - i}\right)^n = -1$, is
    (a) 1 (b) 2
    (c) 3 (d) 4

15. Consider the following statements :
    $S_1 : -8 = 2i \times 4i = \sqrt{(-4)} \times \sqrt{(-16)}$
    $S_2 : \sqrt{(-4)} \times \sqrt{(-16)} = \sqrt{(-4) \times (-16)}$
    $S_3 : \sqrt{(-4) \times (-16)} = \sqrt{64}$
    $S_4 : \sqrt{64} = 8$

of these statements, the *incorrect* one is
(a) S_1 only (b) S_2 only
(c) S_3 only (d) None of these

16. If the multiplicative inverse of a complex number is $(\sqrt{3}+4i)/19$, then the complex number itself is
(a) $\sqrt{3}-4i$ (b) $4+i\sqrt{3}$
(c) $\sqrt{3}+4i$ (d) $4-i\sqrt{3}$

17. If z_1 and $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides whose centre is origin and if $\frac{\text{Im}(z_1)}{\text{Re}(z_1)}=\sqrt{2}-1$, then n is equal to
(a) 8 (b) 16
(c) 24 (d) 32

18. If $|z|=1$, then $\left(\frac{1+z}{1+\bar{z}}\right)$ equals
(a) z (b) $\bar{z}$
(c) z^{-1} (d) None of these

19. For $x_1, x_2, y_1, y_2 \in R$. If $0<x_1<x_2$, $y_1=y_2$ and $z_1=x_1+iy_1$; $z_2=x_2+iy_2$ and $z_3=\frac{1}{2}(z_1+z_2)$, then z_1, z_2 and z_3 satisfy
(a) $|z_1|=|z_2|=|z_3|$
(b) $|z_1|<|z_2|<|z_3|$
(c) $|z_1|>|z_2|>|z_3|$
(d) $|z_1|<|z_3|<|z_2|$

20. Of z is any non-zero complex number then $\arg(z)+\arg(\bar{z})$ is equal to
(a) 0 (b) $\pi/2$
(c) π (d) $3\pi/2$

21. If the following regions in the complex plane, the only one that does not represent a circle, is
(a) $z\bar{z}+i(z-\bar{z})=0$ (b) $\text{Re}\left(\frac{1+z}{1-z}\right)=0$
(c) $\arg\left(\frac{z-i}{z+i}\right)=\frac{\pi}{2}$ (d) $\left|\frac{z-i}{z+1}\right|=1$

22. If $x=\left(\frac{1+i}{2}\right)$, then the expression $2x^4-2x^2+x+3$ equals
(a) $3-(i/2)$ (b) $3+(i/2)$
(c) $(3+i)/2$ (d) $(3-i)/2$

23. If $1, \omega, \omega^2$ are the three cube roots of unity then for $\alpha, \beta, \gamma, \delta \in R$, the expression $\left(\frac{\alpha+\beta\omega+\gamma\omega^2+\delta\omega^2}{\beta+\alpha\omega^2+\gamma\omega+\delta\omega}\right)$ is equal to
(a) 1 (b) ω
(c) $-\omega$ (d) ω^{-1}

24. If ω is a complex cube root of unity and $(1+\omega)^7=A+B\omega$ then, A and B are respectively equal to
(a) 0, 1 (b) 1, 1
(c) 1, 0 (d) -1, 1

25. If 1, ω and ω^2 are the three cube roots of unity, then the roots of the equation $(x-1)^3-8=0$ are
(a) $-1, -1-2\omega, -1+2\omega^2$
(b) $3, 2\omega, 2\omega^2$
(c) $3, 1+2\omega, 1+2\omega^2$
(d) None of these

26. If $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are n, nth roots of unity. the value of $(9-\omega)(9-\omega^2)\ldots(9-\omega^{n-1})$ will be
(a) n (b) 0
(c) $\frac{9^n-1}{8}$ (d) $\frac{9^n+1}{8}$

27. If $8iz^3+12z^2-18z+27i=0$ then
(a) $|z|=3/2$ (b) $|z|=2/3$
(c) $|z|=1$ (d) $|z|=3/4$

28. If $z=re^{i\theta}$, then $|e^{iz}|$ is equal to
(a) $e^{-r\sin\theta}$ (b) $re^{-r\sin\theta}$
(c) $e^{-r\cos\theta}$ (d) $re^{-r\cos\theta}$

29. If z_1, z_2, z_3 are three distinct complex numbers and a, b, c are three positive real numbers such that
$$\frac{a}{|z_2-z_3|}=\frac{b}{|z_3-z_1|}=\frac{c}{|z_1-z_2|}, \text{ then}$$
$$\frac{a^2}{(z_2-z_3)}+\frac{b^2}{(z_3-z_1)}+\frac{c^2}{(z_1-z_2)}=$$
(a) 0 (b) abc
(c) $3abc$ (d) $a+b+c$

30. For all complex numbers z_1, z_2 satisfying $|z_1|=12$ and $|z_2-3-4i|=5$, the minimum value of $|z_1-z_2|$ is
(a) 0 (b) 2
(c) 7 (d) 17

31. If z_1, z_2, z_3 are the vertices of an equilateral triangle in the argand plane, then $(z_1^2 + z_2^2 + z_3^2) = k(z_1z_2 + z_2z_3 + z_3z_1)$ is true for
(a) $k = 1$ (b) $k = 2$
(c) $k = 3$ (d) $k = 4$

32. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is :
(a) of zero area (b) right angled isosceles
(c) equilateral (d) obtuse angled isosceles

33. The value of $\sqrt{i} + \sqrt{(-i)}$ is
(a) 0 (b) $\sqrt{2}$
(c) $-i$ (d) i

34. If z_1, z_2, z_3 are the vertices of an equilateral triangle with centroid z_0, then $z_1^2 + z_2^2 + z_3^2$ equals
(a) z_0^2 (b) $2z_0^2$
(c) $3z_0^2$ (d) $9z_0^2$

35. If a complex number z lies on a circle of radius $1/2$ then the complex number $(-1 + 4z)$ lies on a circle of radius
(a) $1/2$ (b) 1
(c) 2 (d) 4

36. If n is a positive integer but not a multiple of 3 and $z = -1 + i\sqrt{3}$, then $(z^{2n} + 2^n z^n + 2^{2n})$ is equal to
(a) 0 (b) -1
(c) 1 (d) 3×2^n

37. If the vertices of a triangle are $8 + 5i, -3 + i, -2 - 3i$, the modulus and the argument of the complex number representing the centroid of this triangle respectively are
(a) $2, \frac{\lambda}{4}$ (b) $\sqrt{2}, \frac{\pi}{4}$
(c) $2\sqrt{2}, \frac{\pi}{4}$ (d) $2\sqrt{2}, \frac{\pi}{2}$

38. The value of $\sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$ is
(a) -1 (b) 0
(c) $-i$ (d) i

39. If $\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)^{50} = 3^{25}(x - iy)$ where x, y are real, then the ordered pair (x, y) is given by
(a) (0, 3) (b) $(1/2, -\sqrt{3}/2)$
(c) $(-3, 0)$ (d) $(0, -3)$

40. If α, β and γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$ (ω is cube root of unity), then $\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1}$ is
(a) $\frac{3}{\omega}$ (b) ω^2
(c) $2\omega^2$ (d) $3\omega^2$

41. The set of points in an argand diagram which satisfy both $|z| \le 4$ and $\arg z = \pi/3$ is
(a) a circle and a line (b) a radius of a circle
(c) a sector of a circle (d) an infinite part line

42. If the points represented by complex numbers $z_1 = a + ib$, $z_2 = c + id$ and $z_1 - z_2$ are collinear then
(a) $ad + bc = 0$ (b) $ad - bc = 0$
(c) $ab + cd = 0$ (d) $ab - cd = 0$

43. Let A, B and C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then the nine point centre is represented by the complex number
(a) $\frac{z_1 + z_2}{2} - z_3$ (b) $\frac{z_1 + z_2 - z_3}{2}$
(c) $\frac{z_1 + z_2 + z_3}{2}$ (d) $\frac{z_1 - z_2 - z_3}{2}$

44. Let α and β be two distinct complex numbers such that $|\alpha| = |\beta|$. If real part of α is positive and imaginary part of β is negative, then the complex number $(\alpha + \beta)/(\alpha - \beta)$ may be
(a) zero (b) real and negative
(c) real and positive (d) purely imaginary

45. The complex number z satisfies the condition $\left|z - \frac{25}{z}\right| = 24$. The maximum distance from the origin of co-ordinates to the point z is
(a) 25 (b) 30
(c) 32 (d) None of these

46. The points A, B and C represent the complex numbers $z_1, z_2, (1-i)z_1 + iz_2$ respectively on the complex plane. The triangle ABC is
(a) isosceles but not right angled
(b) right angled but not isosceles
(c) isosceles and right angled
(d) None of these

47. The system of equations $\left.\begin{array}{r} |z+1+i| = \sqrt{2} \\ |z| = 3 \end{array}\right\}$ has
(a) no solution (b) one solution
(c) two solutions (d) none of these

48. The centre of the circle represented by $|z+1| = 2|z-1|$ on the complex plane is
(a) 0 (b) 5/3
(c) 1/3 (d) None of these

49. The value of the expression $2(1+\omega)(1+\omega^2) + 3(2\omega+1)(2\omega^2+1) + 4(3\omega+1)(3\omega^2+1) + \ldots + (n+1)(n\omega+1)(n\omega^2+1)$ is
(ω is the cube root of unity)
(a) $\dfrac{n^2(n+1)^2}{4}$ (b) $\left(\dfrac{n(n+1)}{2}\right)^2 + n$
(c) $\left(\dfrac{n(n+1)}{2}\right)^2 - n$ (d) None of these

50. If $\alpha = \cos\left(\dfrac{8\pi}{11}\right) + i\sin\left(\dfrac{8\pi}{11}\right)$ then $\text{Re}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ is
(a) 1/2 (b) – 1/2
(c) 0 (d) None of these

51. If $\sum_{k=0}^{200} i^k + \prod_{p=1}^{50} i^p = x + iy$ then (x, y) is
(a) (0, 1) (b) (1, –1)
(c) (2, 3) (d) (4, 8)

52. If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$ then $|z_1 + z_2 + z_3|$
(a) is less than 6 (b) is more than 3
(c) is less than 12 (d) lies between 6 and 12

53. If $|z| = \max\{|z-1|, |z+1|\}$ then
(a) $|z + \bar{z}| = \frac{1}{2}$ (b) $z + \bar{z} = 1$
(c) $|z + \bar{z}| = 1$ (d) $z - \bar{z} = 5$

54. The equation $|z+i| - |z-i| = k$ represents a hyperbola if
(a) $-2 < k < 2$ (b) $k > 2$
(c) $0 < k < 2$ (d) None of these

55. If $|z - i| \le 2$ and $z_1 = 5 + 3i$ then the maximum value of $|iz + z_1|$ is
(a) $2 + \sqrt{31}$ (b) 7
(c) $\sqrt{31} - 2$ (d) $\sqrt{31} + 2$

56. If $z = \dfrac{\sqrt{3}+i}{2}$ then $(z^{101} + i^{103})^{105}$ equal to
(a) z (b) z^2
(c) z^3 (d) z^4

57. If $|a_k| < 3$, $1 < k < n$, then all the complex numbers z satisfying the equation $1 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$
(a) lie outside the circle $|z| = \frac{1}{4}$
(b) lie inside the circle $|z| = \frac{1}{4}$
(c) lie on the circle $|z| = \frac{1}{4}$
(d) lie in $\frac{1}{3} < |z| < \frac{1}{2}$

58. If X be the set of all complex numbers z such that $|z| = 1$ and define relation R on X by $z_1 R z_2$ is $|\arg z_1 - \arg z_2| = \dfrac{2\pi}{3}$ then R is
(a) Reflexive (b) Symmetric
(c) Transitive (d) Anti-symmetric

59. The roots of the cubic equation $(z + \alpha\beta)^3 = \alpha^3$ $(\alpha \ne 0)$, represent the vertices of a triangle of sides of length
(a) $\dfrac{1}{\sqrt{3}}|\alpha\beta|$ (b) $\sqrt{3}|\alpha|$
(c) $\sqrt{3}|\beta|$ (d) $\dfrac{1}{\sqrt{3}}|\alpha|$

60. The number of points in the complex plane that satisfying the conditions $|z-2| = 2$, $z(1-i) + \bar{z}(1+i) = 4$ is
(a) 0 (b) 1
(c) 2 (d) more than 2

MULTIPLE CHOICE–II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

61. If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$
(a) π (b) $-\pi$
(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

62. If $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the n, nth roots of the unity, then the value of $\sum_{i=0}^{n-1} \frac{\alpha_i}{(3-\alpha_i)}$ is equal to
(a) $\frac{n}{3^n-1}$ (b) $\frac{n-1}{3^n-1}$
(c) $\frac{n+1}{3^n-1}$ (d) $\frac{n+2}{3^n-1}$

63. If z satisfies $|z-1| < |z+3|$ then $\omega = 2z + 3 - i$ satisfies
(a) $|\omega - 5 - i| < |\omega + 3 + i|$
(b) $|\omega - 5| < |\omega + 3|$
(c) $I_m(i\omega) > 1$
(d) $|\arg(\omega - 1)| < \pi/2$

64. ω is a cube root of unity and n is a positive integer satisfying $1 + \omega^n + \omega^{2n} = 0$; then n is of the type
(a) $3m$ (b) $3m+1$
(c) $3m+2$ (d) None of these.

65. If $\frac{z+1}{z+i}$ is a purely imaginary number; then z lies on a
(a) straight line
(b) circle
(c) circle with radius $= 1/\sqrt{2}$
(d) circle passing through the origin.

66. The equation whose roots are nth power of the roots of the equation, $x^2 - 2x\cos\theta + 1 = 0$, is given by
(a) $(x + \cos n\theta)^2 + \sin^2 n\theta = 0$
(b) $(x - \cos n\theta)^2 + \sin^2 n\theta = 0$
(c) $x^2 + 2x\cos n\theta + 1 = 0$
(d) $x^2 - 2x\cos n\theta + 1 = 0$

67. If $|z-1| + |z+3| \le 8$, then the range of values of $|z-4|$ is
(a) $(0, 7)$ (b) $(1, 8)$
(c) $[1, 9]$ (d) $[2, 5]$

68. $\sin^{-1}\left\{\frac{1}{i}(z-1)\right\}$, where z is non real, can be the angle of a triangle if
(a) $\text{Re}(z) = 1$, $I_m(z) = 2$
(b) $\text{Re}(z) = 1$, $-1 \le I_m(z) \le 1$
(c) $\text{Re}(z) + I_m(z) = 0$
(d) None of these

69. If $\frac{\tan\alpha - i(\sin\alpha/2 + \cos\alpha/2)}{1 + 2i\sin\alpha/2}$ is purely imaginary, then α is given by
(a) $n\pi + \pi/4$ (b) $n\pi - \pi/4$
(c) $2n\pi$ (d) $2n\pi + \pi/4$

70. If z_1 and z_2 are non zero complex numbers such that $|z_1 - z_2| = |z_1| + |z_2|$ then
(a) $|\arg z_1 - \arg z_2| = \pi$
(b) $\arg z_1 = \arg z_2$
(c) $z_1 + kz_2 = 0$ for some positive number k
(d) $z_1\bar{z}_2 + \bar{z}_1 z_2 < 0$

71. If z is a complex number and $a_1, a_2, a_3, b_1, b_2, b_3$ all are real then
$$\begin{vmatrix} a_1z + b_1\bar{z} & a_2z + b_2\bar{z} & a_3z + b_3\bar{z} \\ b_1z + a_1\bar{z} & b_2z + a_2\bar{z} & b_3z + a_3\bar{z} \\ b_1z + a_1 & b_2z + a_2 & b_3z + a_3 \end{vmatrix}$$
(a) $(a_1a_2a_3 + b_1b_2b_3)^2 |z|^2$
(b) $|z|^2$
(c) 3
(d) None of these

72. Let
$$A = \frac{2}{\sqrt{3}}e^{i\pi/2},\ B = \frac{2}{\sqrt{3}}e^{-i\pi/6},\ C = \frac{2}{\sqrt{3}}e^{-i5\pi/6}$$
be three points forming a triangle ABC in the argand plane. Then ΔABC is :
(a) equilateral (b) isosceles
(c) scalene (d) None of these

73. If $|z| = 1$, then $\left(\frac{1+z}{1+z}\right)^n + \left(\frac{1+z}{1+z}\right)^n$ is equal to :
(a) $2 \cos n (\arg (z))$
(b) $2 \sin (\arg (z))$
(c) $2 \cos n (\arg (z/2))$
(d) $2 \sin n (\arg (z/2))$

74. If all the roots of $z^3 + az^2 + bz + c = 0$ are of unit modulus, then
(a) $|a| \leq 3$ (b) $|b| > 3$
(c) $|c| < 3$ (d) None of these

75. The trigonometric form of $z = (1 - i \cot 8)^3$ is
(a) $\text{cosec}^3\, 8.\, e^{i\,(24 - 3\pi/2)}$
(b) $\text{cosec}^3\, 8.\, e^{-i\,(24 - 3\pi/2)}$
(c) $\text{cosec}^3\, 8. e^{i\,(36 - \pi/2)}$
(d) $\text{cosec}^2\, 8.\, e^{-24i + \pi/2}$

76. If $1 + \omega + \omega^2 = 0$ then $\omega^{1994} + \omega^{1995}$ is
(a) $-\omega$ (b) $-\omega^2$
(c) $-\omega^3$ (d) $-\omega^4$

77. If $|a_i| < 1$, $\lambda_i > 0$ for $i = 1, 2, 3, .., n$ and $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = 1$, then the value of $|\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n|$ is
(a) $= 1$ (b) < 1
(c) > 1 (d) None of these

78. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then
(a) $\frac{z_1}{z_2}$ is purely real (b) $\frac{z_1}{z_2}$ is purely imaginary
(c) $z_1 \bar{z}_2 + z_2 \bar{z}_1 = 0$ (d) $\text{amp} \frac{z_1}{z_2} = \frac{\pi}{2}$

79. If z_1, z_2, z_3, z_4 are the four complex numbers represented by the vertices of a quadrilateral taken in order such that $z_1 - z_4 = z_2 - z_3$ and $\text{amp}\left(\frac{z_4 - z_1}{z_2 - z_1}\right) = \frac{\pi}{2}$ then the quadrilateral is a
(a) rhombus (b) square
(c) rectangle (d) cyclic quadrilateral

80. Let z_1, z_2 be two complex numbers represented by points on the circle $|z| = 1$ and $|z| = 2$ respectively then
(a) $\max |2z_1 + z_2| = 4$ (b) $\min |z_1 - z_2| = 1$
(c) $\left| z_2 + \frac{1}{z_1} \right| \leq 3$ (d) None of these

81. If α is a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root then
(a) $\alpha + \alpha = 1$ (b) $\alpha + \alpha = 0$
(c) $\alpha + \bar{\alpha} = -1$
(d) The absolute value of the real root is 1

82. If $|z - 3i| = 3$ and $\text{amp}\ z \in \left(0, \frac{\pi}{2}\right)$ then cot $(\text{amp}\ z) - \frac{6}{z}$ is equal to
(a) i (b) 1
(c) -1 (d) $-\frac{1}{i}$

83. Perimeter of the locus represented by arg $\left(\frac{z+i}{z-i}\right) = \frac{\pi}{4}$ is equal to
(a) $\frac{3\pi}{2}$ (b) $\frac{3\pi}{\sqrt{2}}$
(c) $\frac{\pi}{\sqrt{2}}$ (d) None of these

84. The digit in the unit's place in the value of $(739)^{49}$ is
(a) 3 (b) 4
(c) 9 (d) 2

85. If z_1, z_2, z_3, z_4 are roots of the equation
$$a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0,$$
where a_0, a_1, a_2, a_3 and a_4 are real, then
(a) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of the equation
(b) z_1 is equal to at least one of $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$
(c) $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of the equation
(d) None of these

86. If $z_1 \neq -z_2$ and $|z_1 + z_2| = \left|\frac{1}{z_1} + \frac{1}{z_2}\right|$ then
(a) at least one of z_1, z_2 is unimodular
(b) both z_1, z_2 are unimodular
(c) z_1, z_2 is unimodular
(d) None of these

87. If $z = x + iy$ and $\omega = \frac{1 - iz}{z - i}$, then $|\omega| = 1$ implies that in the complex plane.
(a) z lies on imaginary axis
(b) z lies on real axis
(c) z lies on unit circle
(d) none of these

88. If $S_r = \int \sin x \, d(i^r x)$ when $(i = \sqrt{-1})$ then $\sum_{r=1}^{4n-1} S_r$ is $(n \in N)$

(a) $-\cos x + c$ (b) $\cos x + c$
(c) 0 (d) not defined

89. For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ we write $z_1 \cap z_2$ of $x_1 < x_2$ and $y_1 \le y_2$ then for all complex numbers z with $1 \cap z$ we have $\frac{1-z}{1+z} \cap \ldots$ is

(a) 0 (b) 1
(c) 2 (d) 3

90. Let $3 - i$ and $2 + i$ be affixes of two points A and B in the argand plane and P represents of the complex number $z - x + iy$ then the locus of P if $|z - 3 + i| = |z - 2 - i|$ is

(a) Circle on AB as diameter
(b) The line AB
(c) The perpendicular bisector of AB
(d) None of these

Practice Test

MM : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. The number of points in the complex plane that satisfying the conditions $|z - 2| = 2$, $z(1 - i) + \bar{z}(1 + i) = 4$ is

(a) 0 (b) 1
(c) 2 (d) more than 2

2. The distances of the roots of the equation $|\sin\theta_1| z^3 + |\sin\theta_2| z^2 + |\sin\theta_3| z + |\sin\theta_4| = 3$, from $z = 0$, are

(a) greater than 2/3
(b) less than 2/3
(c) greater than $|\sin\theta_1| + |\sin\theta_2| + |\sin\theta_3| + |\sin\theta_4|$
(d) less than $|\sin\theta_1| + |\sin\theta_2| + |\sin\theta_3| + |\sin\theta_4|$

3. The reflection of the complex number $\frac{2-i}{3+i}$ in the straight line $z(1+i) - \bar{z}(i-1)$ is

(a) $\frac{-1-i}{2}$ (b) $\frac{-1+i}{2}$
(c) $\frac{i(i+1)}{2}$ (d) $\frac{-1}{1+i}$

4. Let S be the set of complex number z which satisfy

$\log_{1/3}\{\log_{1/2}(|z|^2 + 4|z| + 3)\} < 0$ then S is

(a) $1 \pm i$ (b) $3 - i$
(c) $\frac{5}{2} + 4i$ (d) Empty set

5. Let $f(z) = \sin z$ and $g(z) = \cos z$. If * denotes a composition of functions then the value of $(f + ig) * (f - ig)$ is

(a) $ie^{e^{iz}}$ (b) $ie^{-e^{iz}}$
(c) $ie^{e^{-iz}}$ (d) $ie^{-e^{-iz}}$

6. If one root of the quadratic equation $(1+i)x^2 - (7+3i)x + (6+8i) = 0$ is $4 - 3i$ then the other root must be

(a) $4 + 3i$ (b) $1 - i$
(c) $1 + i$ (d) $i(1 - i)$

7. Let $f_p(\alpha) = e^{i\alpha/p^2} \cdot e^{2i\alpha/p^2} \cdot e^{3i\alpha/p^2} \ldots e^{i\alpha/p}$ then $\lim_{n \to \infty} f_n(\pi)$ is

(a) 1 (b) -1
(c) i (d) $-i$

8. The common roots of the equations $z^3 + (1+i)z^2 + (1+i)z + i = 0$ and $z^{1993} + z^{1994} + 1 = 0$ are

(a) 1 (b) ω
(c) ω^2 (d) ω^{981}

9. The argument and the principle value of the complex number $\frac{2+i}{4i + (1+i)^2}$ are

(a) $\tan^{-1}(-2)$ (b) $-\tan^{-1} 2$
(c) $\tan^{-1}\left(\frac{1}{2}\right)$ (d) $-\tan^{-1}\left(\frac{1}{2}\right)$

10. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$ then $|9z_1z_2 + 4z_1z_3 + z_3z_2|$ is equal to

(a) 6 (b) 36
(c) 216 (d) None of these

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (d)	**6.** (b)
7. (c)	**8.** (d)	**9.** (d)	**10.** (a)	**11.** (d)	**12.** (d)
13. (c)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (a)
19. (d)	**20.** (a)	**21.** (d)	**22.** (a)	**23.** (b)	**24.** (b)
25. (c)	**26.** (c)	**27.** (a)	**28.** (a)	**29.** (a)	**30.** (b)
31. (a)	**32.** (c)	**33.** (b)	**34.** (c)	**35.** (c)	**36.** (a)
37. (b)	**38.** (d)	**39.** (b)	**40.** (d)	**41.** (c)	**42.** (b)
43. (c)	**44.** (d)	**45.** (a)	**46.** (c)	**47.** (a)	**48.** (b)
49. (b)	**50.** (b)	**51.** (b)	**52.** (c)	**53.** (c)	**54.** (a)
55. (b)	**56.** (c)	**57.** (a)	**58.** (a)	**59.** (b)	**60.** (c)

Multiple Choice –II

61. (a)	**62.** (a)	**63.** (b), (c), (d)	**64.** (b), (c)	**65.** (b), (c), (d)	**66.** (b), (d)
67. (c)	**68.** (b)	**69.** (a), (c), (d)	**70.** (a), (c), (d)	**71.** (d)	**72.** (a)
73. (a)	**74.** (a)	**75.** (a)	**76.** (a), (d)	**77.** (b)	**78.** (b), (c), (d)
79. (c), (d)	**80.** (a), (b), (c)	**81.** (a), (c), (d)	**82.** (a), (d)	**83.** (d)	**84.** (c)
85. (a), b)	**86.** (c)	**87.** (b)	**88.** (b)	**89.** (a)	
90. (c)					

Practice Test

1. (c)	**2.** (a)	**3.** (b), (c), (d)	**4.** (d)	**5.** (d)	**6.** (c), (d)
7. (c)	**8.** (b), (c), (d)	**9.** (a), (b)	**10.** (a)		

2

THEORY OF EQUATIONS

§ 2.1. Quadratic Equation

An equation of the form

$$ax^2 + bx + c = 0, \quad \text{...(1)}$$

where $a, b, c \in c$ and $a \neq 0$, is a *quadratic equation.*

The numbers a, b, c are called the coefficients of this equation.

A root of the quadratic equation (1) is a complex number α such that $a\alpha^2 + b\alpha + c = 0$. Recall that $D = b^2 - 4ac$ is the *discriminant* of the equation (1) and its root are given by the formula.

$$x = \frac{-b + \sqrt{D}}{2a},$$

§ 2.2. Nature of Roots

1. If $a, b, c \in R$ and $a \neq 0$, then
   (a) If $D < 0$, then equation (1) has no roots.
   (b) If $D > 0$, then equation (1) has real and distinct roots, namely,
   $$x_1 = \frac{-b + \sqrt{D}}{2a}, x_2 = \frac{-b - \sqrt{D}}{2a}$$
   and then $$ax^2 + bx + c = a(x - x_1)(x - x_2) \quad \text{...(2)}$$
   (c) If $D = 0$, then equation (1) has real and equal roots
   $$x_1 = x_2 = -\frac{b}{2a}$$
   and then $$ax^2 + bx + c = a(x - x_1)^2. \quad \text{...(3)}$$

To represent the quadratic $ax^2 + bx + c$ in form (2) or (3) is to expand it into linear factors.

2. If $a, b, c \in Q$ and D is a perfect square of a rational number, then the roots are rational and in case it be not a perfect square then the roots are irrational.
3. If $a, b, c \in R$ and $p + iq$ is one root of equation (1) $(q \neq 0)$ then the other must be the conjugate $p - iq$ and vice-versa. $(p, q \in R$ and $i = \sqrt{-1})$.
4. It $a, b, c \in Q$ and $p + \sqrt{q}$ is one root of equation (1) then the other must be the conjugate $p - \sqrt{q}$ and vice-versa. (where p is a rational and $\sqrt{q}$ is a surd).
5. If $a = 1$ and $b, c \in I$ and the root of equation (1) are rational numbers, then these roots must be integers.
6. If equation (1) has more than two roots (complex numbers), then (1) becomes an identity *i.e.,* $$a = b = c = 0$$

§ 2.3. Condition for Common Roots

Consider two quadratic equations :

$$ax^2 + bx + c = 0 \quad \text{and} \quad a'x^2 + b'x + c' = 0$$

(i) If two common roots then

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

(ii) If one common root then

$$(ab' - a'b)(bc' - b'c) = (ca' - c'a)^2.$$

§ 2.4. Location of Roots

(Interval in which roots lie)

Let $f(x) = ax^2 + bx + c = 0$, $a, b, c \in R, a > 0$ and α, β be the roots. Suppose $k, k_1, k_2 \in R$ and $k_1 < k_2$. Then the following hold good :

(i) If both the roots of $f(x) = 0$ are greater than k.
then $D > 0$, $f(k) > 0$ and $-\frac{b}{2a} > k$,

(ii) If both the roots of $f(x) = 0$ are less than k
then $D > 0$, $f(k) > 0$ and $-\frac{b}{2a} < k$,

(iii) If k lies between the roots of $f(x) = 0$,
then $D > 0$ and $f(k) < 0$.

(iv) If exactly one root of $f(x) = 0$ lies in the interval (k_1, k_2)
then $D > 0$, $f(k_1) f(k_2) < 0$,

(v) If both roots of $f(x) = 0$ are confined between k_1 and k_2
then $D > 0$, $f(k_1) > 0$, $f(k_2) > 0$
i.e., $$k_1 < \left(\frac{\alpha + \beta}{2}\right) < k_2 .$$

(vi) **Rolle's theorem :** Let $f(x)$ be a function defined on a closed interval $[a, b]$ such that
(i) $f(x)$ is continuous on $[a, b]$
(ii) $f(x)$ is derivable on (a, b) and
(iii) $f(a) = f(b) = 0$. Then $c \in (a, b)$ s.t. $f'(c) = 0$.

(vii) **Lagrange's theorem :** Let $f(x)$ be a function defined on $[a, b]$, such that
(i) $f(x)$ is continuous on $[a, b]$, and
(ii) $f(x)$ is derivable on (a, b). Then $c \in (a, b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$.

§ 2.5. Wavy Curve Method

(Generalised Method of intervals)

Let $$F(x) = (x - a_1)^{k_1} (x - a_2)^{k_2} (x - a_3)^{k_3} \dots (x - a_{n-1})^{k_{n-1}} (x - a_n)^{k_n} \quad \dots(1)$$
where $k_1, k_2, k_3, \dots, k_n \in N$ and $a_1, a_2, a_3, \dots, a_n$ fixed natural numbers satisfying the condition
$$a_1 < a_2 < a_3 \dots < a_{n-1} < a_n$$

First we mark the numbers $a_1, a_2, \dots, a_n$ on the real axis and the plus sign in the interval of the right of the largest of these numbers, *i.e.*, on the right of a_n. If k_n is even we put plus sign on the left of a_n and if k_n is odd then we put minus sign on the left of a_n. In the next interval we put a sign according to the following rule :

When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number and the polynomial $f(x)$ has same sign if k_{n-1} is an even number. Then we consider the next interval and put a sign in it using the same rule. Thus we consider all the intervals. The solution of $F(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of $F(x) < 0$ is the union of all intervals in which we have put the minus sign.

§ 2.6. Some Important Forms

1. An equation of the form
$$(x - a)(x - b)(x - c)(x - d) = A,$$
where $a < b < c < d$, $b - a = d - c$, can be solved

by a change of variable.

i.e.,
$$y=\frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$$

or
$$y=x-\frac{(a+b+c+d)}{4}$$

2. An equation of the form
$$(x-a)(x-b)(x-c)(x-d)=Ax^2$$
where $ab=cd$, can be reduced to a collection of two quadratic equations by a change of variable $y=x+\frac{ab}{x}$.

3. An equation of the form
$$(x-a)^4+(x-b)^4=A$$
can also be solved by a change of variable, *i.e.*, making a substitution
$$y=\frac{(x-a)+(x-b)}{2}$$

4. The equation of the form
$$|f(x)+g(x)|=|f(x)|+|g(x)|$$
is equivalent of the system
$$f(x)\,g(x)>0$$

5. An equation of the form
$$a^{f(x)}+b^{f(x)}=c$$
where $a, b, c \in R$

and a, b, c satisfies the condition $a^2+b^2=c$

then solution of the equation is $f(x)=2$ and no other solution of this equation.

6. An equation of the form
$$\{f(x)\}^{g(x)}$$
is equivalent to the equation
$$\{f(x)\}^{g(x)}=10^{g(x)\log f(x)} \quad \text{where } f(x)>0$$

§ 2.7. Some Important Results to be Remember

1. $\log_{a^\alpha} b^\beta=\frac{\beta}{\alpha}\log_a b$

2. $f^{\log_a g}=g^{\log_a f}$

3. $a^{\log_a f}=f$

4. $[x+n]=n+[x], n\in I$ when [.] denotes the greatest integer.

5. $x=[x]+\{x\}$, {} denotes the fractional part of x

6. $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{x}\right]=[nx]$

7. $(x)=\begin{cases}[x], & \text{if } 0<\{x\}<\frac{1}{2}\\ [x]+1, & \text{if } \frac{1}{2}<\{x\}<1\end{cases}$

where (x) denotes the nearest integer to x

i.e., $(x)>[x]$

thus $(1{\cdot}3829)=1;\ (1{\cdot}543)=2;\ (3)=3$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate you choice of correct answer for each question by writing one of the letters a, b, c, d which ever is appropriate.

1. Let $f(x) = ax^2 + bx + c$ and $f(-1) < 1$, $f(1) > -1, f(3) < -4$ and $a \neq 0$ then
(a) $a > 0$ (b) $a < 0$
(c) sign of 'a' can not be determined
(d) none of these

2. If α and β are the roots of the equation $x^2 - p(x+1) - q = 0$, then the value of $\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + q} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + q}$ is
(a) 2 (b) 1
(c) 0 (d) None of these

3. If the roots of the equation, $ax^2 + bx + c = 0$, are of the form $\alpha/(\alpha - 1)$ and $(\alpha + 1)/\alpha$, then the value of $(a + b + c)^2$ is
(a) $2b^2 - ac$ (b) $b^2 - 2ac$
(c) $b^2 - 4ac$ (d) $4b^2 - 2ac$

4. The real roots of the equation $5^{\log_5(x^2 - 4x + 5)} = x - 1$ are
(a) 1 and 2 (b) 2 and 3
(c) 3 and 4 (d) 4 and 5

5. The number of real solutions of the equation $2|x|^2 - 5|x| + 2 = 0$ is
(a) 0 (b) 2
(c) 4 (d) infinite

6. The number of real solutions of $x - \frac{1}{x^2 - 4} = 2 - \frac{1}{x^2 - 4}$ is
(a) 0 (b) 1
(c) 2 (d) infinite

7. The number of values of a for which $(a^2 - 3a + 2)x^2 + (a^2 - 5a + 6)x + a^2 - 4 = 0$ is an identity in x is
(a) 0 (b) 1
(c) 2 (d) 3

8. The solution of $x - 1 = (x - [x])(x - \{x\})$ (where $[x]$ and $\{x\}$ are the integral and fractional part of x) is
(a) $x \in R$ (b) $x \in R \sim [1, 2)$
(c) $x \in [1, 2)$ (d) $x \in R \sim [1, 2]$

9. The number of solutions of $2^{\sin(|x|)} = 4^{|\cos x|}$ in $[-\pi, \pi]$ is equal to :
(a) 0 (b) 2
(c) 4 (d) 6

10. The number of values of the triplet (a, b, c) for which
$$a \cos 2x + b \sin^2 x + c = 0$$
is satisfied by all real x is
(a) 2 (b) 4
(c) 6 (d) infinite

11. The coefficient of x in the quadratic equation $ax^2 + bx + c = 0$ was wrongly taken as 17 in place of 13 and its roots were found to be (–2) and (–15). The actual roots of the equation are :
(a) – 2 and 15 (b) – 3 and – 10
(c) – 4 and – 9 (d) – 5 and – 6

12. The value of α for which the equation
$$(\alpha + 5)x^2 - (2\alpha + 1)x + (\alpha - 1) = 0$$
has roots equal in magnitude but opposite in sign, is
(a) 7/4 (b) 1
(c) – 1/2 (d) – 5

13. The number of real solutions of the equation
$$2^{x/2} + (\sqrt{2} + 1)^x = (5 + 2\sqrt{2})^{x/2}$$ is
(a) infinite (b) six
(c) four (d) one

14. The equation $\sqrt{x + 1} - \sqrt{x - 1} = \sqrt{4x - 1}$ has
(a) no solution (b) one solution
(c) two solutions (d) more than two solutions

15. The number of real solutions of the equation $e^x = x$ is
(a) 0 (b) 1
(c) 2 (d) None of these

16. If $\tan\alpha$ and $\tan\beta$ are the roots of the equation $ax^2 + bx + c = 0$ then the value of $\tan(\alpha + \beta)$ is :
(a) $b/(a - c)$ (b) $b/(c - a)$
(c) $a/(b - c)$ (d) $a/(c - a)$

17. If α, β are the roots of the equation $x^2 + x\sqrt{\alpha} + \beta = 0$ then the values of α and β are :
(a) $\alpha = 1$ and $\beta = -1$
(b) $\alpha = 1$ and $\beta = -2$
(c) $\alpha = 2$ and $\beta = 1$
(d) $\alpha = 2$ and $\beta = -2$

18. If α, β are the roots of the equation $8x^2 - 3x + 27 = 0$ then the value of $[(\alpha^2/\beta)^{1/3} + (\beta^2/\alpha)^{1/3}]$ is :
(a) 1/3 (b) 1/4
(c) 1/5 (d) 1/6

19. For $a \neq b$, if the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have a common root, then the value of $(a + b)$ is
(a) -1 (b) 0
(c) 1 (d) 2

20. Let α, β be the roots of the equation $(x - a)(x - b) = c, c \neq 0$. Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are :
(a) a, c (b) b, c
(c) a, b (d) $a + c, b + c$

21. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then the equation whose roots are α^{19} and β^7 is :
(a) $x^2 + x + 1 = 0$ (b) $x^2 - x + 1 = 0$
(c) $x^2 + x + 2 = 0$ (d) $x^2 + 19x + 7 = 0$

22. The number of real solutions of $1 + |e^x - 1| = e^x(e^x - 2)$ is
(a) 0 (b) 1
(c) 2 (d) 4

23. The number of solutions of the equation $|x| = \cos x$ is
(a) one (b) two
(c) three (d) zero

24. The total number of solutions of $\sin \pi x = |\ln|x||$ is
(a) 2 (b) 4
(c) 6 (d) 8

25. The value of p for which both the roots of the equation $4x^2 - 20px + (25p^2 + 15p - 66) = 0$, are less than 2, lies in
(a) $(4/5, 2)$ (b) $(2, \infty)$
(c) $(-1, -4/5)$ (d) $(-\infty, -1)$

26. If the equation $ax^2 + 2bx - 3c = 0$ has non real roots and $(3c/4) < (a + b)$; then c is always
(a) < 0 (b) > 0
(c) ≥ 0 (d) zero

27. The root of the equation $2(1 + i)x^2 - 4(2 - i)x - 5 - 3i = 0$ which has greater modulus is
(a) $(3 - 5i)/2$ (b) $(5 - 3i)/2$
(c) $(3 + i)/2$ (d) $(1 + 3i)/2$

28. Let $a, b, c \in R$ and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation, $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies
(a) $\gamma = \alpha$ (b) $\gamma = \beta$
(c) $\gamma = (\alpha + \beta)/2$ (d) $\alpha < \gamma < \beta$

29. The roots of the equation, $2^{x+2} . 27^{3x/(x-1)} = 9$ are given by
(a) $1 - \log_2 3, 2$
(b) $\log_2 (2/3), 1$
(c) $2, -2$
(d) $-2, 1 - (\log 3)/(\log 2)$

30. The number of real solutions of the equation $\cos(e^x) = 2^x + 2^{-x}$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many

31. If the roots of the equation, $x^2 + 2ax + b = 0$, are real and distinct and they differ by at most $2m$ then b lies in the interval
(a) $(a^2 - m^2, a^2)$ (b) $[a^2 - m^2, a^2)$
(c) $(a^2, a^2 + m^2)$ (d) None of these

32. If $x^2 + px + 1$ is a factor of the expression $ax^3 + bx^2 + c$ then
(a) $a^2 + c^2 = -ab$ (b) $a^2 - c^2 = -ab$
(c) $a^2 - c^2 = -bc$ (d) None of these

33. If a, b, c be positive real numbers, the following system of equations in x, y and z :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, has :

(a) no solution
(b) unique solution
(c) infinitely many solutions
(d) finitely many solutions

34. The number of quadratic equations which remain unchanged by squaring their roots, is
(a) nill (b) two
(c) four (d) infinitely many.

35. If $a<0$, the positive root of the equation $x^2-2a|x-a|-3a^2=0$ is
(a) $a(-1-\sqrt{6})$ (b) $a(1-\sqrt{2})$
(c) $a(-1+\sqrt{6})$ (d) $a(1+\sqrt{2})$

36. If α and β are the roots of $ax^2+bx+c=0$, then the equation $ax^2-bx(x-1)+c(x-1)^2=0$ has roots
(a) $\frac{\alpha}{1-\alpha}, \frac{\beta}{1-\beta}$ (b) $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}$
(c) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (d) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$

37. The solution of the equation $3^{\log_a x}+3x^{\log_a 3}=2$ is given by
(a) $3^{\log_2 a}$ (b) $3^{-\log_2 a}$
(c) $2^{\log_3 a}$ (d) $2^{-\log_3 a}$

38. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+x^2+1=0$ then the equation whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$ is
(a) $(x^2-x+1)^2=0$ (b) $(x^2+x+1)^2=0$
(c) $x^4-x^2+1=0$ (d) $x^2-x+1=0$

39. Given that, for all $x \in R$, The expression $\frac{x^2-2x+4}{x^2+2x+4}$ lies between $\frac{1}{3}$ and 3, the value between which the expression $\frac{9.3^{2x}+6.3^x+4}{9.3^{2x}-6.3^x+4}$ lies are
(a) 3^{-1} and 3 (b) -2 and 0
(c) -1 and 1 (d) 0 and 2

40. The value of $\sqrt{7+\sqrt{7-\sqrt{7+\sqrt{7-\ldots\infty}}}}$ is
(a) 5 (b) 4
(c) 3 (d) 2

41. If x^2+x+1 is a factor of ax^3+bx^2+cx+d then the real root of $ax^3+bx^2+cx+d=0$ is
(a) $-d/a$ (b) d/a
(c) a/d (d) None of these

42. $x^{\log_5 x}>5$ implies
(a) $x \in (0, \infty)$ (b) $x \in \left(0, \frac{1}{5}\right) \cup (5, \infty)$
(c) $x \in (1, \infty)$ (d) $x \in (1, 2)$

43. The values of x which satisfy the eqattion $\sqrt{(5x^2-8x+3)}-\sqrt{(5x^2-9x+4)}=\sqrt{(2x^2-2x)}-\sqrt{(2x^2-3x+1)}$ are
(a) 3 (b) 2
(c) 1 (d) 0

44. The roots of the equation $(a+\sqrt{b})^{x^2-15}+(a-\sqrt{b})^{x^2-15}=2a$, where $a^2-b=1$ are
(a) $\pm 2, \pm\sqrt{3}$ (b) $\pm 4, \pm\sqrt{14}$
(c) $\pm 3, \pm\sqrt{5}$ (d) $\pm 6, \pm\sqrt{20}$

45. The number of number-pairs (x, y) which will satisfy the equation $x^2-xy+y^2=4(x+y-4)$ is
(a) 1 (b) 2
(c) 4 (d) None of these

46. The solution set of the equation $\log_x 2 \log_{2x} 2=\log_{4x} 2$ is
(a) $\{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$ (b) $\left\{\frac{1}{2}, 2\right\}$
(c) $\left\{\frac{1}{4}, 2^2\right\}$ (d) None of these

47. For any real x the expression $2(k-x)[\sqrt{x^2+k^2}]$ can not exceed
(a) k^2 (b) $2k^2$
(c) $3k^2$ (d) None of these

48. The solution of $\left|\frac{x}{x-1}\right|+|x|=\frac{x^2}{|x-1|}$ is
(a) $x \geq 0$ (b) $x>0$
(c) $x \in (1, \infty)$ (d) None of these

49. The number of positive integral solutions of $\frac{x^2(3x-4)^3(x-2)^4}{(x-5)^5(2x-7)^6} \leq 0$ is
(a) Four (b) Three

(c) Two (d) Only one

50. The number of real solutions of the equation $\left(\frac{9}{10}\right)^x = -3 + x - x^2$ is

(a) None (b) One
(c) Two (d) More than two

51. The equation $|x+1|^{\log_{(x+1)}(3+2x-x^2)} = (x-3)|x|$ has

(a) Unique (b) Two solutions
(c) No Solution (d) More than two

52. If $xy = 2(x+y)$, $x < y$ and $x, y \in N$, the number of solutions of the equation

(a) Two
(b) Three
(c) No solutions
(d) Infinitely many solutions

53. The number of real solutions of the system of equations

$$x = \frac{2z^2}{1+z^2},\ y = \frac{2x^2}{1+x^2},\ z = \frac{2y^2}{1+y^2} \text{ is}$$

(a) 1 (b) 2
(c) 3 (d) 4

54. The number of negative integral solutions of $x^2 . 2^{x+1} + 2^{|x-3|+2} = x^2 . 2^{|x-3|+4} + 2^{x-1}$ is

(a) None (b) Only one
(c) Two (d) Four

55. If a be a positive integer, the number of values of a satisfying :

$$\int_0^{\pi/2}\left[a^2\left(\frac{\cos 3x}{4} + \frac{3}{4}\cos x\right) + a\sin x - 20\cos x\right]dx \le -\frac{a^2}{3} \text{ is}$$

(a) Only one (b) Two
(c) Three (d) Four

56. For the equation $|x^2 - 2x - 3| = b$ which statement or statements are true

(a) for $b < 0$ there are no solutions
(b) for $b = 0$ there are three solutions
(c) for $0 < b < 1$ there are four solutions
(d) for $b = 1$ there are two solutions
(e) for $b > 1$ there are no solutions
(f) None of these

57. If $y = 2[x] + 3 = 3[x-2] + 5$, then $[x+y]$ is ($[x]$ denotes the integral part of x)

(a) 10 (b) 15
(c) 12 (d) None of these

58. If α, β, γ are the roots of the equation $x^3 + a_0x^2 + a_1x + a_2 = 0$, then $(1-\alpha^2)(1-\beta^2)(1-\gamma^2)$ is equal to

(a) $(1+a_1)^2 - (a_0+a_2)^2$
(b) $(1+a_1)^2 + (a_0+a_2)^2$
(c) $(1-a_1)^2 + (a_0-a_2)^2$
(d) None of these

59. The roots of the equation $(3-x)^4 + (2-x)^4 = (5-2x)^4$ are

(a) all real
(b) all imaginary
(c) two real & two imaginary
(d) None of these

60. The number of ordered 4-tuple (x, y, z, w) $(x, y, z, w \in [0, 10])$ which satisfies the inequality $2^{\sin^2 x}\, 3^{\cos^2 y}\, 4^{\sin^2 z}\, 5^{\cos^2 w} > 120$ is

(a) 0 (b) 144
(c) 81 (d) Infinite

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

61. The equation $x^{[(\log_3 x)^2 - (9/2)\log_3 x + 5]} = 3\sqrt{3}$ has

(a) at least one real solution
(b) exactly three real solutions
(c) exactly one irrational solution
(d) complex roots

62. Let $f(x)$ be a quadratic expression which is positive for all real x. If $g(x) = f(x) - f'(x) + f''(x)$, then for any real x,

(a) $g(x) > 0$ (b) $g(x) \geq 0$
(c) $g(x) < 0$ (d) $g(x) \leq 0$

63. For $a > 0$, the roots of the equation $\log_{ax} a + \log_x a^2 + \log_{a^2x} a^3 = 0$, are given by

(a) $a^{-4/3}$ (b) $a^{-3/4}$
(c) $a^{-1/2}$ (d) a^{-1}

64. The real values of λ for which the equation, $3x^3 + x^2 - 7x + \lambda = 0$ has two distinct real roots in [0, 1] lie in the interval(s)

(a) $(-2, 0)$ (b) [0, 1]
(c) [1, 2] (d) $(-\infty, \infty)$

65. If α is one root of the equation $4x^2 + 2x - 1 = 0$, then its other root is given by

(a) $4\alpha^3 - 3\alpha$ (b) $4\alpha^3 + 3\alpha$
(c) $\alpha - (1/2)$ (d) $-\alpha - (1/2)$

66. The roots of the equation, $(x^2 + 1)^2 = x(3x^2 + 4x + 3)$, are given by

(a) $2 - \sqrt{3}$ (b) $(-1 + i\sqrt{3})/2$
(c) $2 + \sqrt{3}$ (d) $(-1 - i\sqrt{3})/2$

67. If $2a + 3b + 6c = 0$ $(a, b, c \in R)$ then the quadratic equation $ax^2 + bx + c = 0$ has

(a) At least one in [0, 1]
(b) At least one root in (–1, 1]
(c) At least one root in [0, 2]
(d) None of these

68. Let $F(x)$ be a function defined by $F(x) = x - [x]$, $0 \neq x \in R$, where $[x]$ is the greatest integer less than or equal to x. Then the number of solutions of $F(x) + F(1/x) = 1$ is/are

(a) 0 (b) infinite
(c) 1 (d) 2

69. The largest interval in which $x^{12} - x^9 + x^4 - x + 1 > 0$ is

(a) $[0, \infty)$ (b) $(-\infty, 0]$
(c) $(-\infty, \infty)$ (d) None of these

70. The system of equation $|x - 1| + 3y = 4$, $x - |y - 1| = 2$ has

(a) No solution (b) A unique solution
(c) Two solutions (d) More than two solutions

71. If A, G and H are the Arithmetic mean, Geometric mean and Harmonic mean between two unequal positive integers. Then the equation $Ax^2 - |G|x - H = 0$ has

(a) both roots are fractions
(b) at least one root which is negative fraction
(c) exactly one positive root
(d) at least one root which is an integer

72. If $0 < a < b < c$, and the roots α, β of the equation $ax^2 + bx + c = 0$ are non real complex numbers, then

(a) $|\alpha| = |\beta|$ (b) $|\alpha| > 1$
(c) $|\beta| < 1$ (d) None of these

73. If $5\{x\} = x + [x]$ and $[x] - \{x\} = \frac{1}{2}$ when $\{x\}$ and $[x]$ are fractional and integral part of x then x is

(a) 1/2 (b) 3/2
(c) 5/2 (d) 7/2

74. If α, β, γ are the roots of the equation $x^3 - x - 1 = 0$, then the value of $\Sigma\left(\frac{1+\alpha}{1-\alpha}\right)$ is

(a) – 3 (b) – 5
(c) – 7 (d) None of these

75. If $c > 0$ and the equation $3ax^2 + 4bx + c = 0$ has no real root, then

(a) $2a + c > b$ (b) $a + 2c > b$
(c) $3a + c > 4b$ (d) $a + 3c < b$

76. The solutions of the equation $1! + 2! + 3! + \ldots + (x-1)! + x! = k^2$ and $k \in I$ are

(a) 0 (b) 1
(c) 2 (d) 3

77. If α, β be roots of $x^2 - x - 1 = 0$ and $A_n = \alpha^n + \beta^n$ then A.M. of A_{n-1} and A_n is

(a) $2A_{n+1}$ (b) $1/2\, A_{n+1}$
(c) $2A_{n-2}$ (d) None of these

78. If $a, b, c \in R$ and the equality $ax^2 - bx + c = 0$ has complex roots which are reciprocal of each other then one has

(a) $|b| < |a|$ (b) $|b| < |c|$
(c) $a = c$ (d) $b \leq a$

79. If a, b, c are positive rational numbers such that $a > b > c$ and the quadratic equation $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ has a root in the interval $(-1, 0)$ then

(a) $b + c > a$ (b) $c + a < 2b$
(c) Both roots of the given equation are rational
(d) The equation $ax^2 + 2bx + c = 0$ has both negative real roots

80. Let S be the set of values of 'a' for which 2 lie between the roots of the quadratic equation $x^2 + (a+2)x - (a+3) = 0$ then S is given by
(a) $(-\infty, -5)$ (b) $(5, \infty)$
(c) $(-\infty, -5]$ (d) $[5, \infty)$

81. The values of a for which the equation $2(\log_3 x)^2 - |\log_3 x| + a = 0$ possess four real solutions
(a) $-2 < a < 0$ (b) $0 < a < \frac{1}{8}$
(c) $0 < a < 5$ (d) None of these

82. How many roots does the following equation possess ?

$$3^{|x|}\,|2 - |x|| = 1$$

(a) 1 (b) 2
(c) 3 (d) 4

83. The equation $|x+1||x-1| = a^2 - 2a - 3$ can have real solutions for x if a belongs to
(a) $(-\infty, -1] \cup [3, \infty)$ (b) $[1-\sqrt{5}, 1+\sqrt{5}]$
(c) $[1-\sqrt{5}, -1] \cup [3, 1+\sqrt{5}]$ (d) None of these

84. If a, b, c are rational and no two of them are equal then the equations

$$(b-c)x^2 + (c-a)x + a - b = 0$$

and $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$
(a) have rational roots
(b) will be such that at least one has rational roots
(c) have exactly one root common
(d) have at least one root common

85. A quadratic equation whose roots are $\left(\frac{\gamma}{\alpha}\right)^2$ and $\left(\frac{\beta}{\alpha}\right)^2$, where α, β, γ are the roots of $x^3 + 27 = 0$ is
(a) $x^2 - x + 1 = 0$ (b) $x^2 + 3x + 9 = 0$
(c) $x^2 + x + 1 = 0$ (d) $x^2 - 3x + 9 = 0$

86. The solution set of $(x)^2 + (x+1)^2 = 25$, where (x) is the nearest integer greater than or equal to x, is
(a) $(2, 4)$
(b) $[-5, -4) \cup [2, 3)$
(c) $[-4, -3) \cup [3, 4)$
(d) None of these

87. If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where $[x]$ is the greatest integer less than or equal to x, the number of possible values of x is
(a) 34 (b) 33
(c) 32 (d) None of these

88. If $\sin^x \theta + \cos^x \theta > 1$, $0 < \theta < \pi/2$, then
(a) $x \in [2, \infty)$ (b) $x \in (-\infty, 2]$
(c) $x \in [-1, 1]$ (d) $x \in [-2, 2]$

89. If α, β are roots of $375x^2 - 25x - 2 = 0$ and $s_n = \alpha^n + \beta^n$ then $\lim_{n \to \infty} \sum_{r=1}^{n} s_r$ is
(a) $\frac{7}{116}$ (b) $\frac{1}{12}$
(c) $\frac{29}{358}$ (d) None of these

90. The equation $x^2 + a^2 x + b^2 = 0$ has two roots each of which exceeds a number c, then
(a) $a^4 > 4b^2$
(b) $c^2 + a^2 c + b^2 > 0$
(c) $-a^2/2 > c$
(d) None of these

Practice Test

MM : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. Let α, β, γ be the roots of the equation $(x-a)(x-b)(x-c) = d$, $d \neq 0$, then the roots of the equation $(x-\alpha)(x-\beta)(x-\gamma) + d = 0$ are
(a) a, b, d (b) b, c, d
(c) a, b, c (d) $a+d, b+d, c+d$

2. If one root of the equation $ix^2 - 2(1+i)x + 2 - i = 0$ is $(3-i)$, then the other root is
(a) $3 + i$ (b) $3 + \sqrt{-1}$

(c) $-1+i$ (d) $-1-i$

3. In a quadratic equation with leading coefficient 1, a student reads the coefficient 16 of x wrongly as 19 and obtain the roots are -15 and -4 the correct roots are

(a) 6, 10 (b) $-6, -10$

(c) $-7, -9$ (d) None of these

4. The number of solutions of $|[x] - 2x| = 4$, where $[x]$ denotes the greatest integer $< x$, is

(a) Infinite (b) 4

(c) 3 (d) 2

5. The interval of x in which the inequality $5^{1/4 \log_5^2 x} > 5^{1/5 \log_5 x}$

(a) $(0, 5^{-2\sqrt{5}}]$ (b) $[5^{2\sqrt{5}}, \infty)$

(c) both a and b (d) None of these

6. The solution set of the equation $(x+1)^2 + [x-1]^2 = (x-1)^2 + [x+1]^2$, where $[x]$ and (x) are the greatest integer and nearest integer to x, is

(a) $x \in R$ (b) $x \in N$

(c) $x \in I$ (d) $x \in Q$

7. If $x^2 + px + 1$ is a factor of $2\cos^2\theta\, x^3 + 2x + \sin 2\theta$, then

(a) $\theta = n\pi, n \in I$ (b) $\theta = n\pi + \frac{\pi}{2}, n \in I$

(c) $\theta = 2n\pi, n \in I$ (d) $\theta = \frac{n\pi}{2}, n \in I$

8. If α, β, γ are the roots of the cubic $x^3 + qx + r = 0$, then the value of $\Pi(\alpha - \beta)^2 =$

(a) $-(27q^2 + 4r^3)$ (b) $-(27q + 4r^2)$

(c) $-(27r^2 + 4q^3)$ (d) $-(27r + 4q^2)$

9. The number of real roots of the equation $x^3 + x^2 + 2x + \sin x = 0$ in $[-2\pi, 2\pi]$ is (are)

(a) zero (b) one

(c) two (d) three

10. The number of solutions of the the following inequality

$2^{1/\sin^2 x_2} \cdot 3^{1/\sin^2 x_3} \cdot 4^{1/\sin^2 x_4} \ldots n^{1/\sin^2 x_n} < n!$

where $x_i \in (0, 2\pi)$ for $i = 2, 3, \ldots, n$ is

(a) 1

(b) 2^{n-1}

(c) n^n

(d) infinite number of solutions

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (b)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (c)	**6.** (a)
7. (b)	**8.** (c)	**9.** (c)	**10.** (d)	**11.** (b)	**12.** (c)
13. (d)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (b)
19. (a)	**20.** (c)	**21.** (a)	**22.** (b)	**23.** (b)	**24.** (c)
25. (d)	**26.** (a)	**27.** (a)	**28.** (d)	**29.** (d)	**30.** (a)
31. (b)	**32.** (c)	**33.** (d)	**34.** (c)	**35.** (b)	**36.** (c)
37. (d)	**38.** (b)	**39.** (b)	**40.** (c)	**41.** (a)	**42.** (b)
43. (c)	**44.** (b)	**45.** (a)	**46.** (a)	**47.** (b)	**48.** (c)
49. (b)	**50.** (a)	**51.** (c)	**52.** (a)	**53.** (a)	**54.** (a)
55. (d)	**56.** (a)	**57.** (b)	**58.** (a)	**59.** (c)	**60.** (b)

Multiple Choice –II

61. (a), (b), (c)	**62.** (a), (b)	**63.** (a), (c)	**64.** (a), (b), (d)	**65.** (a), (d)
66. (a), (b), (c), (d)		**67.** (a, b, c)	**68.** (b)	**69.** (c)	**70.** (b)
71. (b), (c)	**72.** (a), (b)	**73.** (b)	**74.** (c)	**75.** (c)	**76.** (b), (d)
77. (b)	**78.** (a), (b), (c)	**79.** (b), (c), (d)	**80.** (a)	**81.** (b)	**82.** (d)
83. (a), (c)	**84.** (a), (c)	**85.** (c)	**86.** (b)	**87.** (c)	**88.** (b)
89. (b)	**90.** (a), (b), (c)				

Practice Test

1. (c)	**2.** (d)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (c)
7. (a)	**8.** (c)	**9.** (b)	**10.** (b)		

3

SEQUENCES AND SERIES

§ 3.1. Arithmetic Progression (A.P.)

(i) If a is the first term and d is the common difference, then A.P. can be written as

$$a,\ a+d,\ a+2d,\ \ldots,\ a+(n-1)\,d,\ \ldots$$

nth term : $T_n = a + (n-1)\,d = l$ (last term)

where $d = T_n - T_{n-1}$

nth term from last $T_n' = l - (n-1)\,d$

(ii) Sum of first n terms : $S_n = \frac{n}{2}\,[2a + (n-1)\,d]$

$$= \frac{n}{2}\,(a + l)$$

and $T_n = S_n - S_{n-1}$

(iii) Arithmetic mean for any n positive numbers $a_1, a_2, a_3, \ldots\ a_n$ is

$$\text{A.M.} = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

(iv) If n arithmetic means $A_1, A_2, A_3, \ldots, A_n$ are inserted between a and b, then

$$A_r = a + \left(\frac{b-a}{n+1}\right) r,\ 1 \le r \le n \text{ and } A_0 = a,\ A_{n+1} = b.$$

§ 3.2. Geometric Progression (G.P.)

(i) If a is the first term and r is the common ratio, then G.P. can be written as $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots$

nth term : $T_n = ar^{n-1} = l$ (last term)

where $r = \frac{T_n}{T_{n-1}}$

nth term from last $T_n' = \frac{l}{r^{n-1}}$

(ii) Sum of first n terms :

$$S_n = \frac{a\,(r^n - 1)}{(r-1)}, \quad \text{if } r > 1$$

and $S_n = \frac{a\,(1 - r^n)}{(1-r)}$, if $r < 1$

$S_n = an$ if $r = 1$

Sum of infinite G.P. when $|\,r\,| < 1$

i.e. $-1 < r < 1$

$\therefore$ $S_\infty = \frac{a}{1-r}$ $(|\,r\,| < 1)$

(iii) Geometric mean for any n positive numbers $b_1, b_2, b_3, \ldots, b_n$ is

$$\text{G.M.} = (b_1 b_2 b_3 \ldots\ b_n)^{1/n}$$

(iv) If n geometric means $G_1, G_2, \ldots, G_n$ are inserted between a and b then

$$G_r = a\left(\frac{b}{a}\right)^{\frac{r}{n+1}}, \ G_0 = a, \text{ and } G_{n+1} = b$$

(v) **To find the value of a recurring decimal :** Let X denote the figure which do not recur, and suppose them x in number; let Y denote the recurring period consisting of y figures. let R denote the value of the recurring decimal;

then $R = 0\,XYYY\ldots.;$

$\therefore$ $10^x \times R = X\cdot YYY\ldots.;$

and $10^{x+y} \times R = XY\cdot YYY\ldots.;$

$\therefore$ therefore by subtraction

$$\therefore \quad R = \frac{XY - X}{(10^{x+y} - 10^x)}$$

§ 3.3. Arithmetic–Geometric Progression (A.G.P.)

(i) If a is the first term, d the common difference and r the common ratio then $a, (a+d)\,r, (a+2d)\,r^2, \ldots, (a+(n-1)\,d)\,r^{n-1}, \ldots.$ is known as A.G.P.

nth term of A.G.P. : $T_n = (a+(n-1)\,d)\,r^{n-1}$

(ii) Sum of first n terms of A.G.P. is

$$S_n = \frac{a}{(1-r)} + \frac{dr\,(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)\,d]\,r^n}{(1-r)}$$

(iii) Sum upto infinite terms of an A.G.P. is

$$S_\infty = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2}. \qquad (|r| < 1)$$

§ 3.4. Natural Numbers

We shall use capital Greek letter Σ (sigma) to denote the sum of series.

(i) $\sum_{r=1}^{n} r = 1+2+3+4+\ldots.+n = \frac{n(n+1)}{2} = \Sigma n$

(ii) $\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\ldots.+n^2 = \frac{n(n+1)(2n+1)}{6} = \Sigma n^2$

(iii) $\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\ldots.+n^3 = \left[\frac{n(n+1)}{2}\right]^2 = [\Sigma n]^2 = \Sigma n^3$

(iv) $\Sigma a = a+a+a+\ldots$ n terms $= na.$

Note : If nth term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$, then

$$S_n = \sum_{r=1}^{n} T_r = a\sum_{r=1}^{n} r^3 + b\sum_{r=1}^{n} r^2 + c\sum_{r=1}^{n} r + \sum_{r=1}^{n} d.$$

§ 3.5. Method of Differences

If the differences of the successive terms of a series are in A.P. or G.P., we can find nth term of the series by the following method :

Step (I) : Denote the nth term and the sum of the series upto n terms of the series by T_n and S_n respectively.

Step (II) : Rewrite the given series with each term shifted by one place to the right.

Step (III) : Subtracting the above two forms of the series, find T_n.

§ 3.6. Harmonic Progression (H.P.)

(i) If the sequence $a_1, a_2, a_3, ..., a_n$ are in H.P. then $\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}$ are in A.P.

n th terms : $T_n = \dfrac{1}{\dfrac{1}{a_1} + (n-1)\left(\dfrac{1}{a_2} - \dfrac{1}{a_1}\right)}$

$= \dfrac{a_1 a_2}{a_2 + (n-1)(a_1 - a_2)}$

(ii) Harmonic mean for n positive numbers $a_1, a_2, a_3, ..., a_n$ is

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right)$$

(iii) If n Harmonic mean $H_1, H_2, H_3, ..., H_n$ are inserted between a and b, then

$$\frac{1}{H_r} = \frac{1}{a} + rd \quad \text{where} \quad d - \frac{(a-b)}{(n+1)\,ab}$$

Theorem : If A, G, H are respectively A.M., G.M., H.M., between a and b both being unequal and positive then

(i) $G^2 = AH$

(ii) $A > G > H$. and every mean must lie between the minimum and the maximum terms. Note that $A = G = H$ iff all terms are equal otherwise $A > G > H$.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If a, b, c are in A.P., then $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in :

(a) A.P. (b) G.P.
(c) H.P. (d) no definite sequence

2. If a, b, c, d, e, f are in A.P., then $(e - c)$ is equal to :

(a) $2(c-a)$ (b) $2(d-b)$
(c) $2(f-d)$ (d) $2(d-c)$

3. If $\log_3 2, \log_3(2^x - 5)$ and $\log_3(2^x - 7/2)$ are in A.P., then the value of x is :

(a) 2 (b) 3
(c) 4 (d) 5

4. If the ratio of the sums of m and n terms of an A.P., is $m^2 : n^2$, then the ratio of its mth and nth terms is :

(a) $(m-1):(n-1)$
(b) $(2m+1):(2n+1)$
(c) $(2m-1):(2n-1)$
(d) none of these

5. If the sum of first n positive integers is $\frac{1}{5}$ times the sum of their squares, then n equals :

(a) 5 (b) 6
(c) 7 (d) 8

6. The interior angles of a polygon are in A. P. the smallest angle is 120° and the common difference is 5°. Then, the number of sides of polygon, is :

(a) 5 (b) 7
(c) 9 (d) 15

7. If a, b, c are in A.P. then the equation $(a-b)x^2 + (c-a)x + (b-c) = 0$ has two roots which are :

(a) rational and equal
(b) rational and distinct
(c) irrational conjugates
(d) complex conjugates

8. If the sum of first n terms of an A.P. is $(Pn + Qn^2)$, where P, Q are real numbers, then the common difference of the A.P., is

(a) $P-Q$ (b) $P+Q$
(c) $2Q$ (d) $2P$

9. If $\left(\dfrac{a^n+b^n}{a^{n-1}+b^{n-1}}\right)$ is the A.M. between a and b, then the value of n is :
(a) -1 (b) 0
(c) $1/2$ (d) 1

10. Given two numbers a and b. Let A denote their single A.M. and S denote the sum of n A.M.'s between a and b then (S/A) depends on :
(a) n, a, b (b) n, a
(c) n, b (d) n only

11. If $x, (2x+2), (3x+3), \ldots$ are in G.P., then the next term of this sequence is :
(a) 27 (b) -27
(c) 13·5 (d) $-13{\cdot}5$

12. If each term of a G.P. is positive and each term is the sum of its two succeeding terms, then the common ratio of the G.P. is :
(a) $\left(\dfrac{\sqrt{5}-1}{2}\right)$ (b) $\left(\dfrac{\sqrt{5}+1}{2}\right)$
(c) $-\left(\dfrac{\sqrt{5}+1}{2}\right)$ (d) $\left(\dfrac{1-\sqrt{5}}{2}\right)$

13. The largest interval for which the series $1+(x-1)+(x-1)^2+\ldots$ ad inf. may be summed, is :
(a) $0<x<1$ (b) $0<x<2$
(c) $-1<x<1$ (d) $-2<x<2$

14. Three numbers, the third of which being 12, form decreasing G.P. If the last term were 9 instead of 12, the three numbers would have formed an A.P. The common ratio of the G.P. is :
(a) 1/3 (b) 2/3
(c) 3/4 (d) 4/5

15. The coefficient of x^{49} in the product $(x-1)(x-3)\ldots(x-99)$ is
(a) -99^2 (b) 1
(c) -2500 (d) None of these

16. If $(1{\cdot}05)^{50}=11{\cdot}658$, then $\sum_{n=1}^{49}(1{\cdot}05)^n$ equals :
(a) 208·34 (b) 212·12
(c) 212·16 (d) 213·16

17. If a, b, c are digits, then the rational number represented by $0{\cdot}cababab\ldots$ is
(a) $cab/990$
(b) $(99c+ab)/990$
(c) $(99c+10a+b)/99$
(d) $(99c+10a+b)/990$

18. If $\log_2(a+b)+\log_2(c+d)\geq 4$. Then the minimum value of the expression $a+b+c+d$ is
(a) 2 (b) 4
(c) 8 (d) None of these

19. The H. M. of two numbers is 4 and their A. M. and G. M. satisfy the relation $2A+G^2=27$, then the numbers are :
(a) -3 and 1 (b) 5 and -25
(c) 5 and 4 (d) 3 and 6

20. If $\Sigma n=55$ then Σn^2 is equal to
(a) 385 (b) 506
(c) 1115 (d) 3025

21. The natural numbers are grouped as follows : {1}, {2, 3, 4}, {5, 6, 7, 8, 9}, ... then the first element of the nth group is :
(a) n^2-1 (b) n^2+1
(c) $(n-1)^2-1$ (d) $(n-1)^2+1$

22. A monkey while trying to reach the top of a pole of height 12 metres takes every time a jump of 2 metres but slips 1 metre while holding the pole. The number of jumps required to reach the top of the pole, is :
(a) 6 (b) 10
(c) 11 (d) 12

23. The sum of the series $1.n+2.(n-1)+3.(n-2)+\ldots+n.1$ is :
(a) $\dfrac{n(n+1)(n+2)}{6}$ (b) $\dfrac{n(n+1)(n+2)}{3}$
(c) $\dfrac{n(n+1)(2n+1)}{6}$ (d) $\dfrac{n(n+1)(2n+1)}{3}$

24. If p, q, r are three positive real numbers are in A.P., then the roots of the quadratic equation : $px^2+qx+r=0$ are all real for :
(a) $\left|\left(\dfrac{r}{p}\right)-7\right|\geq 4\sqrt{3}$ (b) $\left|\dfrac{p}{r}-7\right|<4\sqrt{3}$
(c) all p and r (d) no p and r

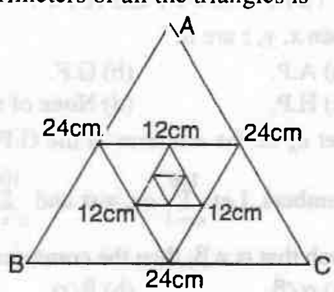
25. If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}\ (x \neq 0)$, then a, b, c, d are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

26. Two A.M.s A_1 and A_2 ; two G.M.s G_1 and G_2 and two H.M.s H_1 and H_2 are inserted between any two numbers : then $H_1^{-1}+H_2^{-1}$ equals
(a) $A_1^{-1}+A_2^{-1}$ (b) $G_1^{-1}+G_2^{-1}$
(c) $G_1G_2/(A_1+A_2)$ (d) $(A_1+A_2)/G_1G_2$.

27. The sum of the products of ten numbers $\pm 1, +2, +3, \pm 4, \pm 5$ taking two at a time is
(a) –55 (b) 55
(c) 165 (d) –165

28. Given that n arithmetic means are inserted between two sets of numbers $a, 2b$ and $2a, b$, where $a, b \in R$. Suppose further that mth mean between these two sets of numbers is same, then the ratio, $a : b$ equals
(a) $n-m+1 : m$ (b) $n-m+1 : n$
(c) $m : n-m+1$ (d) $n : n-m+1$.

29. One side of an equilateral triangle is 24 cm. The mid points of its sides are joined to form another triangle whose mid points are in turn jointed to form still another triangle. This process continues indefinitely. The sum of the perimeters of all the triangles is



(a) 144 cm (b) 169 cm
(c) 400 cm (d) 625 cm

30. If $\frac{a+b}{1-ab}, b, \frac{b+c}{1-bc}$ are in A.P., then $a, \frac{1}{b}, c$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

31. If $a_1, a_2, a_3, \ldots$ are in H.P. and

$f(k) = \sum_{r=1}^{n} a_r - a_k$, then $\frac{a_1}{f(1)}, \frac{a_2}{f(2)}, \frac{a_3}{f(3)}, \ldots, \frac{a_n}{f(n)}$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

32. The sixth term of an A.P. is equal to 2. The value of the common difference of the A.P. which makes the product $a_1a_4a_5$ least is given by
(a) $\frac{8}{5}$ (b) $\frac{5}{4}$
(c) $\frac{2}{3}$ (d) None of these

33. The solution of $\log_{\sqrt{3}} x + \log_{\sqrt[4]{3}} x + \log_{\sqrt[6]{3}} x + \ldots + \log_{\sqrt[16]{3}} x = 36$ is
(a) $x = 3$ (b) $x = 4\sqrt{3}$
(c) $x = 9$ (d) $x = \sqrt{3}$

34. If $a_1, a_2, a_3, .., a_n$ are in H.P. then
$\frac{a_1}{a_2+a_3+\ldots+a_n}, \frac{a_2}{a_1+a_3+\ldots+a_n}, \ldots, \frac{a_n}{a_1+a_2+\ldots+a_{n-1}}$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

35. If the arithmetic progression whose common difference is none zero, the sum of first $3n$ terms is equal to the sum of the next n terms. Then the ratio of the sum of the first $2n$ terms to the next $2n$ terms is
(a) $\frac{1}{5}$ (b) $\frac{2}{3}$
(c) $\frac{3}{4}$ (d) None of these

36. Let a, b, c be three positive prime numbers. The progression in which $\sqrt{a}, \sqrt{b}, \sqrt{c}$ can be three terms (not necessarily consecutive) is
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

37. If n is an odd integer greater than or equal to 1 then the value of $n^3-(n-1)^3+(n-2)^3-\ldots+(-1)^{n-1}1^3$ is

(a) $\dfrac{(n+1)^2(2n-1)}{4}$

(b) $\dfrac{(n-1)^2(2n-1)}{4}$

(c) $\dfrac{(n+1)^2(2n+1)}{4}$

(d) None of these

38. If the sides of a right angled triangle form an A.P. then the sines of the acute angles are

(a) $\dfrac{3}{5}, \dfrac{4}{5}$ (b) $\sqrt{3}, \dfrac{1}{3}$

(c) $\sqrt{\dfrac{\sqrt{5}-1}{2}}, \sqrt{\dfrac{\sqrt{5}+1}{2}}$

(d) $\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}$

39. The sum of n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ is $\dfrac{n(n+1)^2}{2}$ when n is even. When n is odd, the sum is

(a) $\dfrac{n^2(n+1)}{2}$ (b) $\dfrac{n(n^2-1)}{2}$

(c) $2(n+1)^2.(2n+1)$ (d) None of these

40. The coefficient of x^{n-2} in the polynomial $(x-1)(x-2)(x-3)\ldots(x-n)$ is

(a) $\dfrac{n(n^2+2)(3n+1)}{24}$

(b) $\dfrac{n(n^2-1)(3n+2)}{24}$

(c) $\dfrac{n(n^2+1)(3n+4)}{24}$

(d) None of these

41. Let $\{a_n\}$ be a G.P. such that $\dfrac{a_4}{a_6} = \dfrac{1}{4}$ and $a_2 + a_5 = 216$. Then $a_1 =$

(a) 12 or $\dfrac{108}{7}$ (b) 10

(c) 7 or $\dfrac{54}{7}$ (d) None of these

42. If $< a_n >$ is an A.P. and $a_1 + a_4 + a_7 + \ldots + a_{16} = 147$, then $a_1 + a_6 + a_{11} + a_{16} =$

(a) 96 (b) 98

(c) 100 (d) None of these

43. The sum to infinity of the series, $1 + 2\left(1 - \dfrac{1}{n}\right) + 3\left(1 - \dfrac{1}{n}\right)^2 + \ldots$ is

(a) n^2 (b) $n(n+1)$

(c) $n\left(1 + \dfrac{1}{n}\right)^2$ (d) None of these

44. If $a_1, a_2, .., a_n$ are n non zero real numbers such that

$(a_1^2 + a_2^2 + \ldots + a_{n-1}^2)(a_2^2 + a_3^2 + \ldots + a_n^2)$
$< (a_1a_2 + a_2a_3 + \ldots + \ldots + a_{n-1}a_n)^2$

then $a_1, a_2..., a_n$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

45. The cubes of the natural numbers are grouped as $1^3, (2^3, 3^3), (4^3, 5^3, 6^3), \ldots$ then the sum of the numbers in the nth group is

(a) $\frac{1}{8}n^3(n^2+1)(n^2+3)$

(b) $\dfrac{1}{16}n^3(n^2+16)(n^2+12)$

(c) $\dfrac{n^3}{12}(n^2+2)(n^2+4)$

(d) None of these

46. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in A.P. such that

$|a| < 1, |b| < 1$ and $|c| < 1$,

then x, y, z are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

47. Let a_n be the nth term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$, then the common ratio is

(a) α/β (b) β/α

(c) $\sqrt{\alpha/\beta}$ (d) $\sqrt{\beta/\alpha}$

48. If $a_1 = 0$ and $a_1, a_2, a_3, .., a_n$ are real numbers such that $|a_i| = |a_{i-1} + 1|$ for all i then the Arithmetic mean of the numbers $a_1, a_2, .., a_n$ has value x where

(a) $x < -1$ (b) $x < -\dfrac{1}{2}$

(c) $x > -\dfrac{1}{2}$ (d) $x = -\dfrac{1}{2}$

49. If a_1, a_2, a_3 $(a_1 > 0)$ are in G.P. with common ratio r, then the value of r for which the inequality $9a_1 + 5a_3 > 14a_2$ holds can not lie in the interval

(a) $[1, \infty)$ (b) $\left[1, \frac{9}{5}\right]$

(c) $\left[\frac{4}{5}, 1\right]$ (d) $\left[\frac{5}{9}, 1\right]$

50. The coefficient of x^{203} in the expansion of $(x-1)(x^2-2)(x^3-3)\ldots(x^{20}-20)$ is

(a) -35 (b) 21

(c) 13 (d) 15

51. If the sum of n terms of the series $\frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \frac{1+2+3}{1^3+2^3+3^3} + \ldots$ is S_n then S_n exceeds 199 for all n greater than

(a) 99 (b) 50

(c) 199 (d) 100

52. The numbers $3^{2\sin 2x - 1}$, 14, $3^{4 - 2\sin 2x}$ form first three terms of an A.P., its fifth term is equal to

(a) -25 (b) -12

(c) 40 (d) 53

53. If $0{\cdot}2\dot{7}$, x, $0{\cdot}7\dot{2}$, and H.P., then x must be

(a) rational (b) irrational

(c) integer (d) None of these

54. In a sequence of $(4n+1)$ terms the first $(2n+1)$ terms are in A.P. whose common difference is 2, and the last $(2n+1)$ terms are in G.P. whose common ratio is 0·5. If the middle terms of the A.P. and G.P. are equal then the middle term of the sequence is

(a) $\frac{n.\,2^{2n+1}}{2^{2n}-1}$

(b) $\frac{n.\,2^{n+1}}{2^{2n}-1}$

(c) $n.\,2^n$

(d) None of these

55. If $a, a_1, a_2, a_3, \ldots, a_{2n}, b$ are in A.P. and $a, g_1, g_2, g_3, \ldots, g_{2n}, b$ are in G.P. and h is the H. M. of a and b then

$\frac{a_1 + a_{2n}}{g_1 g_{2n}} + \frac{a_2 + a_{2n-1}}{g_2 g_{2n-1}} + \ldots + \frac{a_n + a_{n+1}}{g_n g_{n+1}}$ is equal to

(a) $\frac{2n}{h}$ (b) $2nh$

(c) nh (d) $\frac{n}{h}$

56 It is given that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$ to $\infty = \frac{\pi^4}{90}$ then $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ to ∞ is equal to

(a) $\frac{\pi^4}{96}$ (b) $\frac{\pi^4}{45}$

(c) $\frac{89\pi^4}{90}$ (d) $\frac{\pi^4}{46}$

57. Let $S = \frac{8}{5} + \frac{16}{65} + \ldots + \frac{128}{2^{18}+1}$, then

(a) $S = \frac{1088}{545}$

(b) $S = \frac{545}{1088}$

(c) $S = \frac{1056}{545}$

(d) $S = \frac{545}{1056}$

58. If $\sin\theta$, $\sqrt{2}(\sin\theta + 1)$, $6\sin\theta + 6$ are in G.P. then the fifth term is

(a) 81 (b) $82\sqrt{2}$

(c) 162 (d) None of these

59. If $ln\,(a+c)$, $ln\,(c-a)$, $ln\,(a-2b+c)$ are in A.P., then

(a) a, b, c are in A.P.

(b) a^2, b^2, c^2 are in A.P.

(c) a, b, c are in G.P.

(d) a, b, c are in H.P.

60. Let $a_1, a_2, a_3, \ldots$ be in an A.P. with common difference not a multiple of 3. Then maximum number of consecutive terms so that all are primes is

(a) 2 (b) 3

(c) 5 (d) infinite

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

61. If $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P. and x, y, z are also in A.P. (y being not equal to 0, 1 or –1), then
(a) x, y, z are in G.P.
(b) x, y, z are in H.P.
(c) $x = y = z$
(d) $(x-y)^2 + (y-z)^2 + (z-x)^2 = 0$

62. If d, e, f are in G.P. and the two quadratic equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then
(a) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in H.P.
(b) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in G.P.
(c) $2dbf = aef + cde$
(d) $b^2 df = ace^2$

63. If three unequal numbers p, q, r are in H.P. and their squares are in A.P.; then the ratio $p : q : r$ is
(a) $1 - \sqrt{3} : -2 : 1 + \sqrt{3}$
(b) $1 : \sqrt{2} : -\sqrt{3}$
(c) $1 : -\sqrt{2} : \sqrt{3}$
(d) $1 + \sqrt{3} : -2 : 1 - \sqrt{3}$

64. For a positive integer n, let $\alpha(n) = 1 + 1/2 + 1/3 + 1/4 + \ldots + 1/(2^n - 1)$ then
(a) $\alpha(100) < 100$ (b) $\alpha(100) > 100$
(c) $\alpha(200) < 100$ (d) $\alpha(200) > 100$

65. The pth term T_p of an H.P. is $q(p+q)$ and qth term T_q is $p(p+q)$ when $p > 1, q > 1$ then
(a) $T_{p+q} = pq$ (b) $T_{pq} = p + q$
(c) $T_{p+q} > T_{pq}$ (d) $T_{pq} > T_{p+q}$

66. If the first and $(2n-1)$th terms of an A.P., a G.P. and a H.P. are equal and their nth terms are a, b and c respectively, then
(a) $a = b = c$ (b) $a + c = b$
(c) $a \geq b \geq c$ (d) $ac = b^2$

67. If a, b, c be three unequal positive quantities in H.P. then
(a) $a^{100} + c^{100} > 2b^{100}$
(b) $a^3 + c^3 > 2b^3$
(c) $a^5 + c^5 > 2b^5$
(d) $a^2 + c^2 > 2b^2$

68. The sum of the products taken two at a time of the numbers $1, 2, 2^2, 2^3, .., 2^{n-2}, 2^{n-1}$ is
(a) $\frac{1}{3}2^{2n} + \frac{2}{3}$ (b) $\frac{1}{3}2^{2n} - 2^n + \frac{1}{3}$
(c) $\frac{1}{3}2^{2n} - \frac{1}{3}$ (d) $\frac{1}{3}2^{2n} - 2^n + \frac{2}{3}$

69. The sum of the infinite terms of the sequence $\frac{5}{3^2.7^2} + \frac{9}{7^2.11^2} + \frac{13}{11^2.15^2} + \ldots$ is :
(a) $\frac{1}{18}$ (b) $\frac{1}{36}$
(c) $\frac{1}{54}$ (d) $\frac{1}{72}$

70. The sum of n terms of the series $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \ldots$ is
(a) $\frac{n(n^2+6n+11)}{18(n+1)(n+2)(n+3)}$
(b) $\frac{n^3+6}{18(n+1)(n+2)(n+3)}$
(c) $\frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$
(d) $\frac{1}{6} - \frac{1}{2(n+1)(n+2)(n+3)}$

71. Let a, b, c be positive real numbers, such that $bx^2 + (\sqrt{(a+c)^2 + 4b^2})x + (a+c) > 0 \; \forall \; x \in R$, then a, b, c are in
(a) G. P. (b) A. P.
(c) H. P. (d) None of these

72. $(1\frac{2}{3})^2 + (2\frac{1}{3})^2 + 3^2 + (3\frac{2}{3})^2 + \ldots$ to 10 terms, the sum is :
(a) $\frac{1390}{9}$ (b) $\frac{1790}{9}$
(c) $\frac{1990}{9}$ (d) None of these

73. The consecutive odd integers whose sum is $45^2 - 21^2$ are :

(a) 43, 45, ..., 75 (b) 43, 45, ..., 79
(c) 43, 45, ..., 85 (d) 43, 45, ..., 89

74. If $<a_n>$ and $<b_n>$ be two sequences given by $a_n = (x)^{1/2^n} + (y)^{1/2^n}$ and $b_n = (x)^{1/2^n} - (y)^{1/2^n}$ for all $n \in N$ then $a_1 a_2 a_3 \ldots a_n$ is :

(a) $\frac{x+y}{b_n}$ (b) $\frac{x-y}{b_n}$
(c) $\frac{x^2+y^2}{b_n}$ (d) $\frac{x^2-y^2}{b_n}$

75. If $1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)\,3^a + b}{4}$ then (a, b) is :

(a) $(n-2, 3)$ (b) $(n-1, 3)$
(c) $(n, 3)$ (d) $(n+1, 3)$

76. If $x, |x+1|, |x-1|$ are the three terms of an A.P. its sum upto 20 terms is :

(a) 90 or 175 (b) 180 or 350
(c) 360 or 700 (d) 720 or 1400

77. If $\Sigma n, \frac{\sqrt{10}}{3}\Sigma n^2, \Sigma n^3$ are in G.P. then the value of n is :

(a) 3 (b) 4
(c) 2 (d) non existent

78. If $\sum_{r=1}^{\infty} t_r = \frac{n(n+1)(n+2)(n+3)}{8}$, where t_r denotes the rth term of a series, then $\lim_{n \to \infty} \sum_{r=1}^{\infty} \frac{1}{t_r}$ is :

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$
(c) $\frac{1}{2}$ (d) 1

79. Given that $0 < x < \pi/4$ and $\pi/4 < y < \pi/2$ and $\sum_{k=0}^{\infty} (-1)^k \tan^{2k} x = a$, $\sum_{k=0}^{\infty} (-1)^k \cot^{2k} y = b$, then $\sum_{k=0}^{\infty} \tan^{2k}_x \cot^{2k}_y$ is

(a) $\frac{1}{a} + \frac{1}{b} - \frac{1}{ab}$ (b) $a + b - ab$
(c) $\frac{1}{\frac{1}{a} + \frac{1}{b} - \frac{1}{ab}}$ (d) $\frac{ab}{a+b-1}$

80. If a, b, c are in H.P., then the value of $\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right)$ is

(a) $\frac{2}{bc} - \frac{1}{b^2}$ (b) $\frac{1}{4}\left(\frac{3}{c^2} + \frac{2}{ca} - \frac{1}{a^2}\right)$
(c) $\frac{3}{b^2} - \frac{2}{ab}$ (d) None of these

81. $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 = \ldots$

(a) $\frac{n(n+1)(n+2)}{6}$ (b) Σn^2
(c) ${}^nC_3$ (d) ${}^{n+2}C_3$

82. If an A.P., $a_7 = 9$ if $a_1 a_2 a_7$ is least, the common difference is

(a) $\frac{13}{20}$ (b) $\frac{23}{20}$
(c) $\frac{33}{20}$ (d) $\frac{43}{20}$

83. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in H.P. then $\cos x \sec y/2$ is

(a) 1 (b) $\sqrt{2}$
(c) $\sqrt{3}$ (d) $\sqrt{5}$

84. If 1, $\log_y x$, $\log_z y$, $-15 \log_x z$ are in A.P., then

(a) $z^3 = x$ (b) $x = y^{-1}$
(c) $z^{-3} = y$ (d) $x = y^{-1} = z^3$

85. If b_1, b_2, b_3 $(b_1 > 0)$ are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(a) $r > 3$ (b) $r < 1$
(c) $r = 3{\cdot}5$ (d) $r = 5{\cdot}2$

86. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in G.P., then x is equal to

(a) $\log_a(\log_b a)$
(b) $\log_a(\log_e a) - \log_a(\log_e b)$
(c) $-\log_a(\log_a b)$
(d) $\log_a(\log_e b) - \log_a(\log_e a)$

87. If a, b, c are in H.P., then

(a) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

(b) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$

(c) $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in G.P.

(d) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

88. If the ratio of A. M. between two positive real numbers a and b to their H.M. is $m : n$; then $a : b$ is equal to

(a) $\frac{\sqrt{(m-n)} + \sqrt{n}}{\sqrt{(m-n)} - \sqrt{n}}$

(b) $\frac{\sqrt{n} + \sqrt{(m-n)}}{\sqrt{n} - \sqrt{(m-n)}}$

(c) $\frac{\sqrt{m} + \sqrt{(m-n)}}{\sqrt{m} - \sqrt{(m-n)}}$

(d) $\frac{\sqrt{m} - \sqrt{(m-n)}}{\sqrt{m} + \sqrt{(m-n)}}$

89. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

(a) $a = 1/2$ (b) $b = 8/3$

(c) $c = 9/2$ (d) $e = 0$

90. If 1, $\log_9 (3^{1-x} + 2)$ and $\log_3 (4 \cdot 3^x - 1)$ are in A.P., then x is equal to

(a) $\log_4 3$ (b) $\log_3 4$

(c) $1 - \log_3 4$ (d) $\log_3 (0.75)$

Practice Test

MM : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. If $\sum_{n=1}^{k} \left[\frac{1}{3} + \frac{n}{90}\right] = 21$, where $[x]$ denotes the integral part of x, then $k =$

(a) 84 (b) 80

(c) 85 (d) none of these

2. If $x \in \{1, 2, 3, \ldots, 9\}$ and $f_n(x) = xxx \ldots x$ (n digits), then $f_n^2(3) + f_n(2) =$

(a) $2f_{2n}(1)$

(b) $f_n^2(1)$

(c) $f_{2n}(1)$

(d) $-f_{2n}(4)$

3. In the A.P. whose common difference in non zero, the sum of first $3n$ terms in equal to the sum of next n terms. Then the ratio of the sum of the first $2n$ terms to the next $2n$ terms is :

(a) 1/2 (b) 1/3

(c) 1/4 (d) 1/5

4. If three positive real numbers a, b, c are in A.P. with $abc = 4$, then minimum value of b is

(a) 1 (b) 3

(c) 2 (d) 1/2

5. The series of natural numbers is divided into groups : 1; 2, 3, 4; 5, 6, 7, 8, 9; ... and so on. Then the sum of the numbers in the nth group is

(a) $(2n-1)(n^2 - n + 1)$ (b) $n^3 - 3n^2 + 3n - 1$

(c) $n^3 + (n-1)^3$ (d) $n^3 + (n+1)^3$

6. The numbers of divisons of 1029, 1547 and 122 are in

(a) A.P. (b) G. P.

(c) H.P. (d) none of these

7. The coefficient of x^{15} in the product $(1-x)(1-2x)(1-2^2.x)(1-2^3.x) \ldots (1-2^{15}.x)$ is

(a) $2^{105} - 2^{121}$ (b) $2^{121} - 2^{105}$

(c) $2^{120} - 2^{104}$ (d) $2^{105} - 2^{104}$

8. The roots of equation $x^2 + 2(a-3)x + 9 = 0$ lie between –6 and 1 and 2, $h_1, h_2, \ldots, h_{20}$, $[a]$ are in H.P., where $[a]$ denotes the integral part of a, and 2, $a_1, a_2, \ldots, a_{20}$, $[a]$ are in A.P. then $a_3 h_{18} =$

(a) 6 (b) 12

(c) 3 (d) none of these

9. Value of $L = \lim_{n \to \infty} \frac{1}{n^4}\left[1.\left(\sum_{k=1}^{n} k\right) + 2.\left(\sum_{k=1}^{n-1} k\right) + 3.\left(\sum_{k=1}^{n-2} k\right) + \ldots + n.1\right]$ is

(a) 1/24 (b) 1/12
(c) 1/6 (d) 1/3

10. If $\alpha, \beta, \gamma, \delta$ are in A.P. and $\int_0^2 f(x)\,dx = -4$, where

$$f(x) = \begin{vmatrix} x+\alpha & x+\beta & x+\alpha-\gamma \\ x+\beta & x+\gamma & x-1 \\ x+\gamma & x+\delta & x-\beta+\delta \end{vmatrix}$$

then the common difference d is :

(a) 1 (b) –1
(c) 2 (d) –2

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (c)	**6.** (c)
7. (a)	**8.** (c)	**9.** (d)	**10.** (d)	**11.** (d)	**12.** (a)
13. (b)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (d)	**18.** (c)
19. (d)	**20.** (a)	**21.** (d)	**22.** (c)	**23.** (a)	**24.** (a)
25. (b)	**26.** (d)	**27.** (a)	**28.** (c)	**29.** (a)	**30.** (c)
31. (c)	**32.** (c)	**33.** (d)	**34.** (c)	**35.** (a)	**36.** (d)
37. (a)	**38.** (a)	**39.** (a)	**40.** (b)	**41.** (a)	**42.** (b)
43. (a)	**44.** (b)	**45.** (a)	**46.** (c)	**47.** (a)	**48.** (c)
49. (b)	**50.** (c)	**51.** (c)	**52.** (d)	**53.** (a)	**54.** (a)
55. (a)	**56.** (a)	**57.** (a)	**58.** (c)	**59.** (d)	**60.** (b)

Multiple Choice –II

61. (a), (b), (c), (d)		**62.** (a), (c)	**63.** (a), (d)	**64.** (a), (d)	
65. (a), (b), (c)	**66.** (c), (d)	**67.** (a), (b), (c), (d)		**68.** (d)	**69.** (d)
70. (a), (c)	**71.** (b)	**72.** (d)	**73.** (d)	**74.** (b)	**75.** (d)
76. (b)	**77.** (a)	**78.** (c)	**79.** (c), (d)	**80.** (a), (b), (c)	**81.** (a, d)
82. (c)	**83.** (b)	**84.** (a), (b), (c), (d)		**85.** (a), (b), (c), (d)	
86. (a), (b)	**87.** (a), (b), (c), (d)		**88.** (c)	**89.** (a), (b), (c), (d)	
90. (c, d)					

Practice Test

1. (b)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (a), (c)	**6.** (a)
7. (a)	**8.** (b)	**9.** (a)	**10.** (a), (b)		

4

PERMUTATIONS AND COMBINATIONS

§ 4.1. The number of ways of arranging n distinct objects in a row taking $r\,(0 < r < n)$ at a time is denoted by ${}^nP_r$ or $P(n, r)$.

and
$$ {}^nP_r = n(n-1)(n-2)\ldots(n-r+1) $$
$$ = \frac{n!}{(n-r)!} $$

Note that ${}^nP_0 = 1,\ {}^nP_1 = n$ and ${}^nP_{n-1} = {}^nP_n = n!$

§ 4.2. The number of ways of arranging n distinct objects along a circle is $(n-1)!$

§ 4.3. The number of ways of arranging n beads along a circular wire is $\frac{(n-1)!}{2}$

§ 4.4. The number of permutations of n things taken all at a time, p are alike of one kind, q are alike of another kind and r are alike of a third kind and the rest $n-(p+q+r)$ are all different is
$$ \frac{n!}{p!\,q!\,r!}. $$

§ 4.5. The number of ways of n distinct objects taking r of them at a time where any object may be repeated any number of times is n^r.

§ 4.6. The number of ways of selecting $r\,(o < r < n)$ objects out of n distinct objects is denoted by ${}^nC_r$ or $C(n, r)$ and
$$ {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,.\,(r-1)\,.\,(r-2)\ldots 2\cdot 1} $$
$$ = \frac{n!}{r!\,(n-r)!} = \frac{{}^nP_r}{r!} $$

if $r > n$, then ${}^nC_r = 0$

§ 4.7. The number of selecting at least one object out of "n" distinct objects $= 2^n - 1$

§ 4.8. The number of combinations of r things $(r < n)$ out of n identical things is 1.

§ 4.9. The number of selecting r objects from n alike objects $= n + 1$ (where $r = 0, 1, 2, \ldots, n$)

§ 4.10. The number of combinations of n distinct objects taken r at a time, when k particular objects always occur is ${}^{n-k}C_{r-k}$. If k particular objects never occur, then the number of combinations of n distinct objects taken r at time is ${}^{n-k}C_r$.

§ 4.11. If out of $(p+q+r+s)$ things, p are alike of one kind, and q are alike of second kind, r are like of third kind and the rest are different, then the total number of combinations is $(p+1)(q+1)(r+1)\,2^s - 1$

§ 4.12. The number of factors of $n = m_1^{\alpha_1}\, m_2^{\alpha_2} \ldots m_k^{\alpha_k}$

where ($m_1, m_2, \ldots, m_k$ are different primers) is
$$ (\alpha+1)(\alpha_2+1)\ldots(\alpha_k+1). $$

§ 4.13. Division into groups

(i) The number of ways in which $(p+q+r)$ elements be divided into three groups of p, q, r elements respectively

$$= \frac{(p+q+r)!}{p!\,q!\,r!}$$

(ii) If $3p$ distinct elements are divided in three groups each containing p elements, then number of divisions $= \dfrac{3p!}{3!\,(p!)^3}$

(iii) If $3p$ distinct elements are divided equally among three persons, then the number of divisions

$$= \frac{3p!}{(p!)^3}.$$

§ 4.14. Arrangements in groups

(i) The number of ways in which n different things can be arranged in r different groups $n!\,.\,{}^{n-1}C_{r-1}$.

(ii) The number of ways in which n things of the same kind can be distributed into r different groups is ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$, according as groups or are not permitted.

§ 4.15. Derangements : The number of derangements (No object goes to its scheduled place) of n objects

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!}\right)$$

§ 4.16. Some Important Results

(i) The coefficient of x^r in the expansion of $(1-x)^{-n} = {}^{n+r-1}C_r$.

(ii) If there are k objects of one kind, l objects of another kind and so on; then the number of ways of choosing r objects out of these objects (*i.e.*, $l + k + \dots$) is

The coefficient of x^r in $(1 + x + x^2 + x^3 + \dots + x^k)\,(1 + x + x^2 + \dots + x^l)\dots$ Further if one object of each kind is to be included, then the number of ways of choosing r objects is

The coefficient of x^r in $(x + x^2 + \dots + x^k)\,(x + x^2 + \dots + x^l)\dots$

(iii) If there are k object of one kind, l object of another kind and so on; then the number of possible arrangements/permutations of r objects out of these objects (*i.e.*, $l + k + \dots$) is coefficient of x^r in

$$r!\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^l}{l!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^k}{k!}\right)\dots$$

(iv) How to find number of solutions of the equation :
If the equation

$$\alpha + 2\beta + 3\gamma + \dots + q\,\theta = n \qquad \dots (1)$$

(a) If zero included then number of solutions of (1)

$$= \text{coefficient of } x^n \text{ in } (1-x)^{-1}\,(1-x^2)^{-1}\,(1-x^3)^{-1}\dots(1-x^q)^{-1}$$

(b) If zero excluded then number of solutions of (1)

$$= \text{coefficient of } x^{n-\frac{q(q+1)}{2}} \text{ in } (1-x)^{-1}\,(1-x^2)^{-1}\,(1-x^3)^{-1}\dots(1-x^q)^{-1}$$

§ 4.17. Number of Rectangles and Squares

(i) Number of retangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$

and number of squares of any size is $\sum_{r=1}^{n} r^2$

(ii) In a rectangle of $n \times p$ $(n < p)$ number of rectangle of any size is $\dfrac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n} (n+1-r)(p+1-r)$.

§ 4.18. Some Important Results to Remember

1. The number of ways in which n different things can be distributed into r different groups is $r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \dots + (-1)^{n-1}\,{}^rC_{r-1}$

or

Coefficient of x^n in $n!\,(e^x - 1)^r$.
Here blank groups are not allowed.

2. Exponent of prime p in $n!$ is

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots + \left[\frac{n}{p^s}\right]$$

where s is the largest natural number such that $p^s \le n < p^{s+1}$ and [.] denotes the greatest integer.

3. n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then the number of parts into which these lines divide the plane is equal to $1 + \Sigma n$

4. The sum of the digits in the unit place of all numbers formed with the help of $a_1, a_2, \dots, a_n$ taken all at a time is

$$= (n-1)!\,(a_1 + a_2 + \dots + a_n)$$

5. The sum of all digit numbers that can be formed using the digits $a_1, a_2, \dots a_n$ (repitition of digits not allowed) is

$$= (n-1)!\,(a_1 + a_2 + \dots + a_n)\left(\frac{(10^n - 1)}{9}\right)$$

6. If there are n rows, I row has α_1 squares, II row has α_2 squares, III row has α_3 squares, and so on. If we placed β X s in the squares such that each row contain at least one X, the number of ways

= coefficient of x^β in

$$({}^{\alpha_1}C_1\,x + {}^{\alpha_1}C_2\,x^2 + \dots + {}^{\alpha_1}C_{\alpha_1}\,x^{\alpha_1}) \times ({}^{\alpha_2}C_1\,x + {}^{\alpha_2}C_2\,x^2 + \dots + {}^{\alpha_2}C_{\alpha_2}\,x^{\alpha_2})$$

$$\times ({}^{\alpha_3}C_1\,x + {}^{\alpha_3}C_2\,x^2 + \dots + {}^{\alpha_3}C_{\alpha_3}\,x^{\alpha_3}) \times \dots$$

7. Given n distinct points in a plane, no three of which are collinear then the number of line segments they determine is ${}^nC_2$

In particular : The number of diagonals in n-gon (n sides closed polygon) is

$${}^nC_2 - n$$

If in which m points are collinear ($m > 3$) then the number of line segment is

$$({}^nC_2 - {}^mC_2) + 1$$

8. Given n distinct points in a plane, no three of which are collinear then the number of triangles is ${}^nC_3$

If in which m points are collinear ($m > 3$) then the number of triangles is

$${}^nC_3 - {}^mC_3$$

9. Given n distinct points on the circumference of a circle, then

(i) Number of straight lines = ${}^nC_2$

(ii) Number of triangles = ${}^nC_3$

(iii) Number of quadrilaterals = ${}^nC_4$

(iv) Number of pentagon = ${}^nC_5$ etc.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. When simplified, the expression $^{47}C_4 + \sum_{n=1}^{5} {}^{52-n}C_3$ equals
   (a) $^{47}C_5$ (b) $^{49}C_4$
   (c) $^{52}C_5$ (d) $^{52}C_4$
2. If $^nC_{r-1} = 10$, $^nC_r = 45$ and $^nC_{r+1} = 120$ then r equals
   (a) 1 (b) 2
   (c) 3 (d) 4
3. The least positive integral value of x which satisfies the inequality $^{10}C_{x-1} > 2 \cdot {}^{10}C_x$ is
   (a) 7 (b) 8
   (c) 9 (d) 10
4. The number of diagonals that can be drawn in an octagon is
   (a) 16 (b) 20
   (c) 28 (d) 40
5. The number of triangles that can be formed joining the angular points of decagon, is
   (a) 30 (b) 45
   (c) 90 (d) 120
6. If n is an integer between 0 and 21, then the minimum value of $n!\,(21-n)!$ is
   (a) $9!\,2!$ (b) $10!\,11!$
   (c) $20!$ (d) $21!$
7. The maximum number of points of intersection of 8 circles, is
   (a) 16 (b) 24
   (c) 28 (d) 56
8. The maximum number of points of intersection of 8 straight lines, is
   (a) 8 (b) 16
   (c) 28 (d) 56
9. The maximum number of points into which 4 circles and 4 straight lines intersect, is
   (a) 26 (b) 50
   (c) 56 (d) 72
10. If 7 points out of 12 lie on the same straight line than the number of triangles thus formed, is
    (a) 19 (b) 185
    (c) 201 (d) 205
11. The total number of ways in which 9 different toys can be distributed among three different children so that the youngest gets 4, the middle gets 3 and the oldest gets 2, is
    (a) 137 (b) 236
    (c) 1240 (d) 1260
12. Every one of the 10 available lamps can be switched on to illuminate certain Hall. The total number of ways in which the hall can be illuminated, is
    (a) 55 (b) 1023
    (c) 2^{10} (d) $10!$
13. The number of ways in which 7 persons can be seated at a round table if two particular persons are not to sit together, is :
    (a) 120 (b) 480
    (c) 600 (d) 720
14. The number of ways in which r letters can be posted in n letter boxes in a town, is :
    (a) n^r (b) r^n
    (c) ^nP_r (d) ^nC_r
15. The number of ways in which three students of a class may be assigned a grade of *A*, *B*, *C* or *D* so that no two students receive the same grade, is :
    (a) 3^4 (b) 4^3
    (c) $^4P_3$ (d) $^4C_3$
16. The number of ways in which the letters of the word ARRANGE can be made such that both R's do not come together is :
    (a) 900 (b) 1080
    (c) 1260 (d) 1620
17. Six identical coins are arranged in a row. The total number of ways in which the number of heads is equal to the number of tails, is

(a) 9 (b) 20
(c) 40 (d) 120

18. If 5 parallel straight lines are intersected by 4 parallel straight lines, then the number of parallelograms thus formed, is :
(a) 20 (b) 60
(c) 101 (d) 126

19. The total number of numbers that can be formed by using all the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places, is
(a) 3 (b) 6
(c) 9 (d) 18

20. The sides AB, BC and CA of a triangle ABC have **3**, 4 and **5** interior points respectively on them. The number of triangles that can be constructed using these interior points as vertices, is
(a) 205 (b) 208
(c) 220 (d) 380

21. Total number of words formed by using 2 vowels and 3 consonents taken from 4 vowels and 5 consonents is equal to
(a) 60 (b) 120
(c) 720 (d) None of these

22. Ten different letters of an alphabet are given. Words with five letters (not necessarily meaningful or pronounceable) are formed from these letters. The total number of words which have *atleast* one letter repeated, is
(a) 21672 (b) 30240
(c) 69760 (d) 99748

23. A 5-digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways, this can be done, is
(a) 216 (b) 240
(c) 600 (d) 720

24. Twenty eight matches were played in a football tournament. Each team met its opponent only once. The number of teams that took part in the tournament, is
(a) 7 (b) 8
(c) 14 (d) None of these

25. Everybody in a room shakes hand with everybody else. The total numebr of handshakes is 66. The total number of persons in the room is
(a) 11 (b) 12
(c) 13 (d) 14

26. The total number of 3-digit *even* numbers that can be composed from the digits 1, 2, 3, ..., 9, when the repetition of digits *is not* allowed, is
(a) 224 (b) 280
(c) 324 (d) 405

27. The total number of 5-digit telephone numbers that can be composed with distinct digits, is
(a) ${}^{10}P_2$ (b) ${}^{10}P_5$
(c) ${}^{10}C_5$ (d) None of these

28. A car will hold 2 persons in the front seat and 1 in the rear seat. If among 6 persons only 2 can drive, the number of ways, in which the car can be filled, is
(a) 10 (b) 18
(c) 20 (d) 40

29. In an examination there are three multiple choice questions and each question has 4 choices of answers in which only one is correct. The total number of ways in which an examinee can fail to get all answers correct is
(a) 11 (b) 12
(c) 27 (d) 63

30. The sum of the digits in the unit's place of all the numbers formed with the digits 5, 6, 7, 8 when taken all at a time, is
(a) 104 (b) 126
(c) 127 (d) 156

31. Two straight lines intersect at a point O. Points $A_1, A_2, .., A_n$ are taken on one line and points $B_1, B_2, ..., B_n$ on the other. If the point O is not to be used, the number of triangles that can be drawn using these points as vertices, is :
(a) $n(n-1)$ (b) $n(n-1)^2$
(c) $n^2(n-1)$ (d) $n^2(n-1)^2$

32. How many different nine digit numbers can be formed from the number 22 33 55 888 by

rearranging its digits so that the odd digits occupy even positions ?

(a) 16 (b) 36
(c) 60 (d) 180

33. For $2 < r < n, \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$

(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$
(c) $2\binom{n+r}{r}$ (d) $\binom{n+2}{r}$

34. The number of positive integers satisfying the inequality

$${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} < 100 \text{ is}$$

(a) Nine (b) Eight
(c) Five (d) None of these

35. A class has 21 students. The class teacher has been asked to make n groups of r students each and go to zoo taking one group at a time. The size of group (*i.e.*, the value of r) for which the teacher goes to the maximum number of times is (no group can go to the zoo twice)

(a) 9 or 10 (b) 10 or 11
(c) 11 or 12 (d) 12 or 13

36. The number of ways in which a score of 11 can be made from a through by three persons, each throwing a single die once, is

(a) 45 (b) 18
(c) 27 (d) 68

37. The number of positive integers with the property that they can be expressed as the sum of the cubes of 2 positive integers in two different way is

(a) 1 (b) 100
(c) infinite (d) 0

38. The number of triangles whose vertices are the vertices of an octagon but none of whose sides happen to come from the octagon is

(a) 16 (b) 28
(c) 56 (d) 70

39. There are n different books and p copies of each in a library. The number of ways in which one or more than one book can be selected is

(a) $p^n + 1$ (b) $(p+1)^n - 1$
(c) $(p+1)^n - p$ (d) p^n

40. In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no lines passes through both points A and B, and no two are parallel, then the number of intersection points the lines have is equal to

(a) 535 (b) 601
(c) 728 (d) 963

41. We are required to form different words with the help of the letters of the word INTEGER. Let m_1 be the number of words in which I and N are never together and m_2 be the number of words which being with I and end with R, then m_1/m_2 is given by

(a) 42 (b) 30
(c) 6 (d) 1/30

42. If a denotes the number of permutations of $x + 2$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x - 11$ things taken all at a time such that $a = 182\,bc$, then the value of x is

(a) 15 (b) 12
(c) 10 (d) 18

43. There are n points in a plane of which no three are in a straight line except 'm' which are all in a straight line. Then the number of different quadrilaterals, that can be formed with the given points as vertices, is

(a) ${}^{n}C_4 - {}^{m}C_3\,{}^{n-m+1}C_1 - {}^{m}C_4$
(b) ${}^{n}C_4 - {}^{m}C_3\,{}^{n-m}C_1 + {}^{m}C_4$
(c) ${}^{n}C_4 - {}^{m}C_3\,({}^{m-n}C_1) - {}^{n}C_4$
(d) ${}^{n}C_4 + {}^{n}C_3 \cdot {}^{m}C_1$

44. The number of ordered triples of positive integers which are solutions of the equation $x + y + z = 100$ is

(a) 5081 (b) 6005
(c) 4851 (d) 4987

45. The number of numbers less than 1000 that can be formed out of the digits 0, 1, 2, 4 and 5, no digit being repeated, is

(a) 69 (b) 68
(c) 130 (d) None of these

46. A is a set containing n elements. A subset P_1 is chosen, and A is reconstructed by replacing the elements of P_1. The same process is repeated for subsets $P_1, P_2, \ldots P_m$ with $m > 1$. The number of ways of choosing $P_1, P_2, \ldots, P_m$ so that $P_1 \cup P_2 \cup \ldots \cup P_m = A$ is

(a) $(2^m - 1)^{mn}$ (b) $(2^n - 1)^m$
(c) ${}^{m+n}C_m$ (d) None of these

47. On a railway there are 20 stations. The number of different tickets required in order that it may be possible to travel from every station to every station is

(a) 210 (b) 225
(c) 196 (d) 105

48. A set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the element of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is

(a) $2^{2n} - {}^{2n}C_n$ (b) 2^n
(c) $2^n - 1$ (d) 3^n

49. A father with 8 children takes 3 at a time to the zoological Gardens, as often as he can without taking the same 3 children together more than once. The number of times he will go to the garden is

(a) 336 (b) 112
(c) 56 (d) None of these

50. If the $(n+1)$ numbers $a, b, c, d, ..,$ be all different and each of them a prime number, then the number of different factors (other than 1) of $a^m . b . c . d \ldots$ is

(a) $m - 2^n$ (b) $(m+1)\, 2^n$
(c) $(m+1)\, 2^n - 1$ (d) None of these

51. The numebr of selections of four letters from the letters of the word ASSASSINATION is

(a) 72 (b) 71
(c) 66 (d) 52

52. The number of divisors a number 38808 can have, excluding 1 and the number itself is :

(a) 70 (b)72
(c) 71 (d) None of these

53. The letters of the word SURITI are written in all possible orders and these words are written out as in a dictionary. Then the rank of the word SURITI is

(a) 236 (b) 245
(c) 307 (d) 315

54. The total number of seven-digit numbers then sum of whose digits is even is

(a) 9×10^6 (b) 45×10^5
(c) 81×10^5 (d) 9×10^5

55. In a steamer there are stalls for 12 animals and there are cows, horses and calves (not less than 12 of each) ready to be shipped; the total number of ways in which the shipload can be made is

(a) 3^{12} (b) 12^3
(c) ${}^{12}P_3$ (d) ${}^{12}C_3$

56. The number of non-negative integral solution of $x_1 + x_2 + x_3 + 4x_4 = 20$ is

(a) 530 (b) 532
(c) 534 (d) 536

57. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 and 7 so that digits do not repeat and the terminal digits are even is

(a) 144 (b) 72
(c) 288 (d) 720

58. Given that n is the odd, the number of ways in which three numbers in A.P. can be selected from 1, 2, 3, 4, .., n is

(a) $\frac{(n-1)^2}{2}$ (b) $\frac{(n+1)^2}{2}$
(c) $\frac{(n+1)^2}{2}$ (d) $\frac{(n-1)^2}{4}$

59. A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q$ contains exactly two elements is

(a) $9 . {}^nC_2$ (b) $3^n - {}^nC_2$
(c) $2 . {}^nC_n$ (d) None of these

60. The number of times the digit 5 will be written when listing the integers from 1 to 1000 is

(a) 271 (b) 272
(c) 300 (d) None of these

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer(s). for each question, write the letters a, b, c, d corresponding to the correct answer(s).

61. Eight straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. The number of parts into which these lines divide the plane, is

(a) 29 (b) 32
(c) 36 (d) 37

62. The number of ways of painting the faces of a cube with six different glours is

(a) 1 (b) 6
(c) 6 ! (d) ${}^6C_6$

63. Number of divisors of the form $4n+2$ $(n>0)$ of the integer 240 is

(a) 4 (b) 8
(c) 10 (d) 3

64. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 & 7. The smallest value of n for which this is possible is :

(a) 6 (b) 7
(c) 8 (d) 9

65. The position vector of a point P is $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, when $x, y, z \in N$ and $\vec{a}=\hat{i}+\hat{j}+\hat{k}$. If $\vec{r}.\vec{a}=10$, The number of possible position of P is

(a) 36 (b) 72
(c) 66 (d) ${}^9C_2$

66. Sanjay has 10 friends among whom two are married to each other. She wishes to invite 5 of the them for a party. If the married couple refuse to attend separately then the number of different ways in which she can invite five friends is

(a) ${}^8C_5$ (b) $2\times{}^8C_3$
(c) ${}^{10}C_5-2\times{}^8C_4$ (d) None of these

67. There are n seats round a table marked 1, 2, 3, ..., n. The number of ways in which $m\,(<n)$ persons can take seats is

(a) ${}^nP_m$ (b) ${}^nC_m\times(m-1)\,!$
(c) ${}^nC_m\times m\,!$ (d) ${}^{n-1}P_{m-1}$

68. If a, b, c, d are odd natural numbers such that $a+b+c+d=20$ then the number of values of the ordered quadruplet (a, b, c, d) is

(a) 165 (b) 310
(c) 295 (d) 398

69. The numebr of rectangles excluding squares from a rectangle of size 15×10 is :

(a) 3940 (b) 4940
(c) 5940 (d) 6940

70. In a certain test, there are n questions. In this test 2^{n-i} students gave wrong answers to at least i questions, where $i=1, 2, 3, .., n$. If the total number of wrong answers given is 2047, then n is equal to

(a) 10 (b) 11
(c) 12 (d) 13

71. The exponent of 3 in 100 ! is

(a) 12 (b) 24
(c) 48 (d) 96

72. The number of integral solutions of $x_1+x_2+x_3=0$ with $x_i>-5$ is

(a) 34 (b) 68
(c) 136 (d) 500

73. The number of ways in which 10 candidates $A_1, A_2, ..., A_{10}$ can be ranked so that A_1 is always above A_2 is

(a) $\frac{10\,!}{2}$ (b) $8\,!\times{}^{10}C_2$
(c) ${}^{10}P_2$ (d) ${}^{10}C_2$

74. If all permutations of the letters of the word AGAIN are arranged as in dictionary, then fiftieth word is

(a) NAAGI (b) NAGAI
(c) NAAIG (d) NAIAG

75. In a class tournament when the participants were to play one game with another, two class players fell ill, having played 3 games each. If the total numebr of games played is 84, the number of participants at the beggining was

(a) 15 (b) 30

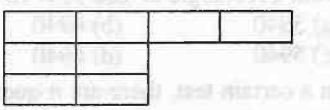
(c) ${}^6C_2$ (d) 48

76. The number of ways of distributing 10 different books among 4 students $(S_1 - S_4)$ such that S_1 and S_2 get 2 books each and S_3 and S_4 get 3 books each is

(a) 12600 (b) 25200

(c) ${}^{10}C_4$ (d) $\frac{10\,!}{2\,!\,2\,!\,3\,!\,3\,!}$

77. The number of different ways the letters of the word VECTOR can be placed in the 8 boxes of the given below such that no row empty is equal to



(a) 26 (b) $26 \times 6\,!$

(c) $6\,!$ (d) $2\,! \times 6\,!$

78. In the next world cup of cricket there will be 12 teams, divided equally in two groups. Teams of each group will play a match against each other. From each group 3 top teams will qualify for the next round. In this round each team will play against others once. Four top teams of this round will qualify for the semifinal round, when each team will play against the others once. Two top teams of this round will go to the final round, where they will play the best of three matches. The minimum number of matches in the next world cup will be

(a) 54 (b) 53

(c) 52 (d) None of these

79. Two lines intersect at O. Points $A_1, A_2, .., A_n$ are taken on one of them and $B_1, B_2, .., B_n$ on the other the number of triangles that can be drawn with the help of these $(2n + 1)$ points is

(a) n (b) n^2

(c) n^3 (d) n^4

80. Seven different lecturers are to deliver lectures in seven periods of a class on a particular day. A, B and C are three of the lectures. The number of ways in which a routine for the day can be made such that A delivers his lecture before B, and B before C, is

(a) 210 (b) 420

(c) 840 (d) None of these

81. If 33 ! is divisible by 2^n then the maximum value of $n =$

(a) 33 (b) 32 (c) 31 (d) 30

82. The number of zeros at the end of 100 ! is

(a) 54 (b) 58

(c) 24 (d) 47

83. The maximum number of different permutations of 4 letters of the word EARTHQUAKE is

(a) 1045 (b) 2190

(c) 4380 (d) 2348

84. In a city no persons have identical set of teeth and there is no person without a tooth. Also no person has more than 32 teeth. If we disregard the shape and size of tooth and consider only the positioning of the teeth, then the maximum population of the city is

(a) 2^{32} (b) $2^{32} - 1$

(c) $2^{32} - 2$ (d) $2^{32} - 3$

85. Ten persons, amongst whom are A, B & C are speak at a function. The number of ways in which it can be done if A wants to speak before B, and B wants to speak before C is

(a) $\frac{10\,!}{6}$ (b) 21870

(c) $\frac{10\,!}{3\,!}$ (d) ${}^{10}P_7$

86. The number of ways in which a mixed double game can be arranged from amongst 9 married couples if no husband and wife play in the same game is

(a) 756 (b) 1512

(c) 3024 (d) None of these

87. In a college examination, a candidate is required to answer 6 out of 10 questions which are divided into two sections each containing 5 questions. further the candidate

is not permitted to attempt more than 4 questions from either of the section. The number of ways in which he can make up a choice of 6 questions is

(a) 200 (b) 150

(c) 100 (d) 50

88. The number of ways in which 9 identical balls can be placed in three identical boxes is

(a) 55 (b) $\frac{9\,!}{(3\,!)^4}$

(c) $\frac{9\,!}{(3\,!)^3}$ (d) 12

89. If the number of arrangements of $(n-1)$ things taken from n different things is k times the number of arrangements of $n-1$ things taken from n things in which two things are identical then the value of k is

(a) $1/2$ (b) 2

(c) 4 (d) None of these

90. The number of different seven digit numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number is

(a) ${}^7P_2\, 2^5$ (b) ${}^7C_2\, 2^5$

(c) ${}^7C_2\, 5^2$ (d) None of these

Practice Test

MM : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[$10 \times 2 = 20$]

1. The number of points (x, y, z) is space, whose each co-ordinate is a negative integer such that $x + y + z + 12 = 0$ is

(a) 385 (b) 55

(c) 110 (d) None of these

2. The number of divisors of $2^2 \cdot 3^3 \cdot 5^3 \cdot 7^5$ of the form $4n + 1$, $n \in N$ is

(a) 46 (b) 47

(c) 96 (d) 94

3. The number of ways in which 30 coins of one rupee each be given to six persons so that none of them receives less than 4 rupees is

(a) 231 (b) 462

(c) 693 (d) 924

4. The number of integral solutions of the equation $2x + 2y + z = 20$ where $x \geq 0, y \geq 0$ and $z > 0$ is

(a) 132 (b) 11

(c) 33 (d) 66

5. The number of ways to select 2 numbers from {0, 1, 2, 3, 4} such that the sum of the squares of the selected numbers is divisible by 5 are (repitition of digits is allowed).

(a) ${}^9C_1$ (b) ${}^9P_8$

(c) 9 (d) 7

6. The number of ways in which we can choose 2 distinct integers from 1 to 100 such that difference between them is at most 10 is

(a) ${}^{10}C_2$ (b) 72

(c) ${}^{100}C_2 - {}^{90}C_2$ (d) None of these

7. Number of points having position vector $a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$ where $a, b, c \in \{1, 2, 3, 4, 5\}$ such that $2^a + 3^b + 5^c$ is divisible by 4 is

(a) 70 (b) 140

(c) 210 (d) 280

8. If a be an element of the set $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and α, β, γ are integers such that $\alpha\beta\gamma = a$, then the number of positive integral solutions of $\alpha\beta\gamma = a$ is

(a) 32 (b) 48

(c) 64 (d) 80

9. If n objects are arranged in a row, then the number of ways of selecting three of these objects so that no two of them are next to each other is

(a) $\frac{(n-2)(n-3)(n-4)}{6}$ (b) ${}^{n-2}C_3$

(c) ${}^{n-3}C_3 + {}^{n-3}C_2$ (d) None of these

10. Number of positive integral solutions of $abc = 30$ is

(a) 9 (b) 27

(c) 81 (d) 243

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (d)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (d)	**6.** (b)
7. (d)	**8.** (c)	**9.** (b)	**10.** (b)	**11.** (d)	**12.** (b)
13. (b)	**14.** (a)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (b)
19. (d)	**20.** (a)	**21.** (d)	**22.** (c)	**23.** (a)	**24.** (b)
25. (b)	**26.** (a)	**27.** (d)	**28.** (d)	**29.** (d)	**30.** (d)
31. (c)	**32** (c)	**33.** (d)	**34.** (b)	**35.** (b)	**36.** (c)
37. (c)	**38.** (a)	**39.** (b)	**40.** (a)	**41.** (b)	**42.** (b)
43. (c)	**44.** (c)	**45.** (b)	**46.** (d)	**47.** (a)	**48.** (d)
49. (c)	**50.** (c)	**51.** (a)	**52.** (a)	**53.** (a)	**54.** (b)
55. (a)	**56.** (d)	**57.** (d)	**58.** (d)	**59.** (d)	**60.** (c)

Multiple Choice –II

61. (d)	**62.** (a), (d)	**63.** (a)	**64.** (b)	**65.** (a), (d)	**66.** (b), (c)
67. (a), (c)	**68.** (a)	**69.** (c)	**70.** (b)	**71.** (c)	**72.** (c)
73. (a), (b)	**74.** (c)	**75.** (a), (c)	**76.** (b), (d)	**77.** (b)	**78.** (b)
79. (c)	**80.** (c)	**81.** (c)	**82.** (c)	**83.** (b)	**84.** (b)
85. (a), (c), (d)	**86.** (b)	**87.** (a)	**88.** (d)	**89.** (b)	**90.** (b)

Practice Test

1. (b)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (a), (b), (c)	**6.** (c)
7. (a)	**8.** (c)	**9.** (a), (b), (c)	**10.** (b)		

BINOMIAL THEOREM

§ 5.1. Binomial Theorem (for a positive integral index)

If n is a positive integer and $x, y \in C$ then

$$(x+y)^n = {}^nC_0\, x^{n-0}\, y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \dots + {}^nC_n\, y^n$$

Here ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are called binomial coefficients.

§ 5.2. Some Important Points to Remember

(i) The number of terms in the expansion are $(n+1)$.

(ii) **General term :**

General term $= (r+1)th$ term

$\therefore \Rightarrow T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$, where $r = 0, 1, 2, \dots, n$.

(iii) **Middle term :** The middle term depends upon the value of n.

(a) If n is even, then total no. of term in the expansion is odd. So there is only one middle term

i.e., $\left(\frac{n}{2}+1\right)$th term is the middle term.

(b) If n is odd, then total number of terms in the expansion is even. So there are two middle terms

i.e., $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th are two middle terms.

(iv) **To find $(p+1)$ th term from end :**

$(p+1)$ th term from end $=$ $(n-p+1)$ th term from beginning

$= T_{n-p+1}$

(v) **Greatest Term :**

To find the greatest term (numerically) in the expansion of $(1+x)^n$.

(a) Calculate $p = \left|\frac{x(n+1)}{(x+1)}\right|$

(b) If p is integer, then T_p and T_{p+1} are equal and both are greatest term.

(c) If p is not integer. Then $T_{[p]+1}$ is the greatest term. Where $[\cdot]$ denotes the greatest integral part.

How to find greatest term in the expansion of $(x+y)^n$:

$$(x+y)^n = x^n\left(1+\frac{y}{x}\right)^n$$

then find the greatest term in $(1+y/x)^n$.

(vi) **Greatest Coefficient:**

(a) If n is even, then greatest coefficient $= {}^nC_{n/2}$

(b) If n is odd, then greatest coefficients are ${}^nC_{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}$

(vii) **Important Formulae:**

(a) $C_0 + C_1 + C_2 + C_3 + \dots + C_n = 2^n$

(b) $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

$\Rightarrow$ Sum of odd binomial coefficients = Sum of even binomial coefficients.

(c) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n$

(d) $C_0C_r + C_1C_{r+1} + C_2\,C_{r+2} + \ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$

where $C_0, C_1, C_2, C_3, \ldots\ldots$ represent ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \ldots\ldots$

(viii) **An Important Theorem :**

If $(\sqrt{P} + Q)^n = I + f$ where I and n are positive integers, n being odd, and $0 < f < 1$, then show that $(I + f)\,f = k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof. Given $\sqrt{P} - Q < 1$

$\therefore \quad 0 < (\sqrt{P} - Q)^n < 1$

Now let $\quad (\sqrt{P} - Q)^n = f'$ where $0 < f' < 1$.

$\therefore \quad I + f - f' = (\sqrt{P} + Q)^n - (\sqrt{P} - Q)^n$

$\because$ R.H.S. contains even powers of $\sqrt{P}$ ($\because n$ is odd)

Hence R.H.S. and I are integers.

$\therefore \quad f - f'$ is also integer.

$\therefore \quad \Rightarrow f - f' = 0 \qquad \because -1 < f - f' < 1$

or $\quad f = f'$

$\therefore \quad (I + f)\,f = (I + f)\,f' = (\sqrt{P} + Q)^n (\sqrt{P} - Q)^n = (P - Q^2)^n = k^n.$

Note. If n is even integer then

$$(\sqrt{P} + Q)^n + (\sqrt{P} - Q)^n = I + f + f'$$

Hence L.H.S. and I are integers.

$\therefore \quad f + f'$ is also integer.

$\Rightarrow \quad f + f' = 1 \qquad \because 0 < f + f' < 2$

$\therefore \quad f' = (1 - f)$

Hence $\quad (I + f)(1 - f) = (I + f)\,f' = (\sqrt{P} + Q)^n (\sqrt{P} - Q)^n$

$= (P - Q^2)^n = k^n.$

(ix) **Multinomial Expansion**

If $n \in N$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is

$\dfrac{n!}{a_1!\,a_2!\,a_3!\ldots a_k!}\,x_1^{a_1}\,.\,x_2^{a_2}\,x_3^{a_3}\ldots\ldots x_k^{a_k}$, where $a_1 + a_2 + a_3 + \ldots + a_k = n$

and $0 < a_i < n$, $i = 1, 2, 3, \ldots k$. and the number of terms in the expansion are ${}^{n+k-1}C_{k-1}$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. The coefficients of x^2y^2, yzt^2 and $xyzt$ in the expansion of $(x + y + z + t)^4$ are in the ratio

(a) 4 : 2 : 1 (b) 1 : 2 : 4

(c) 2 : 4 : 1 (d) 1 : 4 : 2

2. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification, is

(a) 50 (b) 51

(c) 154 (d) 202

3. If the coefficient of x^3 in the expansion of $(1 + ax)^4$ is 32, then a equals

(a) 2 (b) 3

(c) 4 (d) 6

4. In the expansion of $(1 + x)^{43}$, the coefficients of the $(2r + 1)$th and the $(r + 2)$th terms are equal, then the value of r, is

(a) 14 (b) 15

(c) 16 (d) 17

5. If the three successive coefficients in the Binomial expansion of $(1+x)^n$ are 28, 56 and 70 respectively, then n equals

(a) 4 (b) 6
(c) 8 (d) 10

6. If m and n are any two odd positive integers with $n < m$ then the largest positive integer which divides all numbers of the form $(m^2 - n^2)$, is

(a) 4 (b) 6
(c) 8 (d) 9

7. The number $5^{25} - 3^{25}$ is divisible by

(a) 2 (b) 3
(c) 5 (d) 7

8. For a positive integer n, if the expansion of $(2x^{-1} + x^2)^n$ has a term independent of x, then a possible value for n is

(a) 10 b) 16
(c) 18 (d) 22

9. The term independent of x in the expansion of $\left(\sqrt{\left(\frac{x}{3}\right)} + \sqrt{\frac{3}{2x^2}}\right)^{10}$ is

(a) 5/12 (b) 1
(c) ${}^{10}C_1$ (d) None of these

10. The term independent of x in the expansion of

$$[(t^{-1} - 1)\,x + (t^{-1} + 1)^{-1}\,x^{-1}]^8 \text{ is}$$

(a) $56\left(\frac{1-t}{1+t}\right)^3$ (b) $56\left(\frac{1+t}{1-t}\right)^3$
(c) $70\left(\frac{1-t}{1+t}\right)^4$ (d) $70\left(\frac{1+t}{1-t}\right)^4$

11. If $(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$ then $a_0 + a_2 + a_4 + \ldots + a_{2n}$ equals

(a) $\frac{1}{2}(3^n + 1)$ (b) $\frac{1}{2}(3^n - 1)$
(c) $\frac{1}{2}(1 - 3^n)$ (d) $\frac{1}{2} + 3^n$

12. If the sum of the binomial coefficients in the expansion of $\left(x + \frac{1}{x}\right)^n$ is 64, then the term independent of x is equal to

(a) 10 (b) 20
(c) 40 (d) 60

13. The sum of coefficients of the two middle terms in the expansion of $(1+x)^{2n-1}$ is equal to

(a) ${}^{(2n-1)}C_n$ (b) ${}^{(2n-1)}C_{n+1}$
(c) ${}^{2n}C_{n-1}$ (d) ${}^{2n}C_n$

14. If C_r stands for ${}^nC_r$ then the sum of first $(n+1)$ terms of the series $aC_0 + (a+d)\,C_1 + (a+2d)\,C_2 + \ldots$, is

(a) 0 (b) $[a + nd]\,2^n$
(c) $[2a + (n-1)\,d]\,2^{n-1}$ (d) $[2a + nd]\,2^{n-1}$

15. The number of rational terms in $(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{5})^{10}$ is

(a) 6 (b) 4
(c) 3 (d) 1

16. If the number of terms in the expansion of $(1 + 2x - 3x^2)^n$ is 36, then n equals :

(a) 7 (b) 8
(c) 9 (d) none of these

17. If the sum of the coefficients in the expansion of $(2 + 3cx + c^2x^2)^{12}$ vanishes then c equals

(a) –1, 2 (b) 1, 2
(c) 1, –2 (d) –1, –2

18. If the sum of odd numbered terms and the sum of even numbered terms in the expansion of $(x+a)^n$ are A and B respectively then the value of $(x^2 - a^2)^n$ is :

(a) $4\,AB$ (b) $A^2 - B^2$
(c) $A^2 + B^2$ (d) None of these

19. The largest term in the expansion of $(2+3x)^{25}$ where $x = 2$ is its

(a) 13th term (b) 19th term
(c) 20th term (d) 26th term

20. The sum of the series

$$\frac{1}{1!\,(n-1)!} + \frac{1}{3!\,(n-3)!} + \frac{1}{5!\,(n-5)!} + \ldots + \frac{1}{(n-1)!\,1!} \text{ is}$$

(a) $\frac{2^{n-1}}{(n-1)!}$ (b) $\frac{2^n}{(n-1)!}$
(c) $\frac{2^{n-1}}{n!}$ (d) $\frac{2^n}{n!}$

21. The greatest coefficient in the expansion of $(1+x)^{2n+2}$ is

(a) $\dfrac{(2n)!}{(n!)^2}$ (b) $\dfrac{(2n+2)!}{\{(n+1)!\}^2}$

(c) $\dfrac{(2n+2)!}{n!\,(n+1)!}$ (d) $\dfrac{(2n)!}{n!\,(n+1)!}$

22. For integer $n > 1$, the digit at unit place in the number $\sum_{r=0}^{100} r! + 2^{2^n}$ is

(a) 4 (b) 3

(c) 1 (d) 0

23. If $C_0, C_1, C_2, \ldots, C_n$ are the Binomial coefficients in the expansion of $(1+x)^n$, n being even, then

$C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \ldots + (C_0 + C_1 + C_2 + \ldots + C_{n-1})$ is equal to

(a) $n\,2^n$ (b) $n.\,2^{n-1}$

(c) $n.\,2^{n-2}$ (d) $n.\,2^{n-3}$

24. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then the value of

$$C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + \ldots + \frac{1}{(n+1)}C_n \text{ is}$$

(a) $\dfrac{2^{n-1}}{(n+1)}$ (b) $\dfrac{2^{n+1}}{(n+1)}$

(c) $\dfrac{2^{n-1}-1}{(n+1)}$ (d) $\dfrac{2^{n+1}-1}{(n+1)}$

25. If the seventh terms from the beginning and the end in the expansion of $\left(\sqrt[3]{2} + \dfrac{1}{\sqrt[3]{3}}\right)^n$ are equal then n equals

(a) 9 (b) 12

(c) 15 (d) 18

26. The expression $({}^{10}C_0)^2 - ({}^{10}C_1)^2 + \ldots - ({}^{10}C_9)^2 + ({}^{10}C_{10})^2$ equals

(a) ${}^{10}C_5$ (b) $-\,{}^{10}C_5$

(c) $({}^{10}C_5)^2$ (d) $(10!)^2$

27. The number of terms in the expansion of $(\sqrt{3} + \sqrt[4]{5})^{124}$ which are integers, is equal to

(a) nil (b) 30

(c) 31 (d) 32

28. The expression ${}^nC_0 + 4.\,{}^nC_1 + 4^2.\,{}^nC_2 + \ldots + 4^n.\,{}^nC_n$, equals

(a) 2^{2n} (b) 2^{3n}

(c) 5^n (d) None of these

29. The first integral term in the expansion of $(\sqrt{3} + \sqrt[3]{2})^9$, is its

(a) 2nd term (b) 3rd term

(c) 4th term (d) 5th term

30. The number of rational terms in the expansion of $(1 + \sqrt{2} + \sqrt[3]{3})^6$ is

(a) 6 (b) 7

(c) 5 (d) 8

31. The coefficient of $a^2 b^4 c$ in the expansion of $(1 + a + b - c)^9$ is

(a) $2.\,{}^9C_7.\,{}^7C_4$ (b) $-2.\,{}^9C_2.\,{}^7C_3$

(c) ${}^9C_7.\,{}^7C_4$ (d) None of these

32. The greatest value of the term independent of x in the expansion of $(x \sin\alpha + x^{-1}\cos\alpha)^{10}$, $\alpha \in R$ is

(a) 2^5 (b) $\dfrac{10!}{(5!)^2}$

(c) $\dfrac{1}{2^5} \cdot \dfrac{10!}{(5!)^2}$ (d) None of these

33. If $(1 + x + 2x^2)^{20} = a_0 + a_1 x + a_2 x^2 + \ldots + a_{40} x^{40}$ then $a_0 + a_2 + a_4 \ldots + a_{38}$ equals

(a) $2^{19}(2^{20}+1)$ (b) $2^{19}(2^{20}-1)$

(c) $2^{20}(2^{19}-1)$ (d) None of these

34. If 7 divides $32^{32^{32}}$, the remainder is

(a) 1 (b) 0

(c) 4 (d) 6

35. $\sum_{r=0}^{n-1} \dfrac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}$ is equal to

(a) $\dfrac{n}{2}$ (b) $\dfrac{n+1}{2}$

(c) $\dfrac{n(n+1)}{2}$ (d) $\dfrac{n(n-1)}{2(n+1)}$

36. The largest term in the expansion of $\left(\dfrac{b}{2} + \dfrac{b}{2}\right)^{100}$ is

(a) b^{100} (b) $\left(\dfrac{b}{2}\right)^{100}$

(c) ${}^{100}C_{50}\left(\frac{b}{2}\right)^{100}$ (d) None of these

37. The coefficient of x^n is the polynomial $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)(x + {}^{2n+1}C_2)\ldots(x + {}^{2n+1}C_n)$ is

(a) 2^{n+1} (b) $2^{2n+1} - 1$
(c) 2^{2n} (d) None of these

38. If the fourth term of $\left(\sqrt{x^{\left(\frac{1}{1+\log_{10} x}\right)}} + {}^{12}\sqrt{x}\right)^6$ is equal to 200 and $x > 1$, then x is equal to

(a) $10\sqrt{2}$ (b) 10
(c) 10^4 (d) None of these

39. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then the value of $\sum_{k=0}^{n} (k+1)^2 C_k$ is

(a) $2^{n-3}(n^2 + 5n + 4)$
(b) $2^{n-2}(n^2 + 5n + 4)$
(c) $2^{n-2}(5n + 4)$
(d) None of these

40. If $\{x\}$ denotes the fractional part of x, then $\left\{\frac{3^{2n}}{8}\right\}$, $n \in N$ is

(a) 3/8 (b) 7/8
(c) 1/8 (d) None of these

41. The sum of the last ten coefficients in the expansion of $(1+x)^{19}$ when expanded in ascending powers of x is

(a) 2^{18} (b) 2^{19}
(c) $2^{18} - {}^{19}C_{10}$ (d) None of these

42. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals

(a) $(n-1)\,a_n$ (b) $n\,a_n$
(c) $\frac{1}{2}n\,a_n$ (d) None of these

43. The coefficient of x^m in $(1+x)^m + (1+x)^{m+1} + \ldots + (1+x)^n$, $m < n$ is

(a) ${}^{n+1}C_{m+1}$ (b) ${}^{n-1}C_{m-1}$
(c) ${}^nC_m$ (d) ${}^nC_{m+1}$

44. The last two digits of the number 3^{400} are

(a) 39 (b) 29
(c) 01 (d) 43

45. The unit digit of $17^{1983} + 11^{1983} - 7^{1983}$ is

(a) 1 (b) 2
(c) 3 (d) 0

46. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1 x + a_2 x^2 + \ldots + a_{20} x^{20}$ then a_1 equals

(a) 10 (b) 20
(c) 210 (d) 420

47. In the expansion of $(1+x)^n (1+y)^n (1+z)^n$, the sum of the coefficients of the terms of degree r is

(a) $({}^nC_r)^3$ (b) $3.\,{}^nC_r$
(c) ${}^{3n}C_r$ (d) ${}^nC_{3r}$

48. If the second term in the expansion $\left({}^{13}\sqrt{a} + \frac{a}{\sqrt{a^{-1}}}\right)^n$ is $14a^{5/2}$, then the value of ${}^nC_3/{}^nC_2$ is

(a) 4 (b) 3
(c) 12 (d) 6

49. Which of the following expansion will have term containing x^2

(a) $(x^{-1/5} + 2x^{3/5})^{25}$ (b) $(x^{3/5} + 2x^{-1/5})^{24}$
(c) $(x^{3/5} - 2x^{-1/5})^{23}$ (d) $(x^{3/5} + 2x^{-1/5})^{22}$

50. Coefficient of $1/x$ in the expansion of $(1+x)^n (1 + 1/x)^n$ is

(a) $\frac{n!}{(n-1)!\,(n+1)!}$
(b) $\frac{2n!}{(n-1)!\,(n+1)}$
(c) $\frac{n!}{(2n-1)!\,(2n+1)!}$
(d) $\frac{2n!}{(2n-1)\,(2n+1)!}$

51. The coefficient of x^{53} in the expansion $\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} \cdot 2^m$ is

(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$
(c) $-{}^{100}C_{53}$ (d) $-{}^{100}C_{100}$

52. The value of x, for which the 6th term in the expansion of

$$\left\{2^{\log_2 \sqrt{(9^{x-1}+7)}} + \frac{1}{2^{(1/5)\log_2 (3^{x-1}+1)}}\right\}^7$$ is 84 is equal to

(a) 4 (b) 3

(c) 2 (d) 5

53. $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\, {}^rC_p\, 2^p\right)$ is equal to

(a) $4^n - 3^n + 1$ (b) $4^n - 3^n - 1$

(c) $4^n - 3^n + 2$ (d) $4^n - 3^n$

54. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then $\sum\sum_{0<i<j<n} C_i C_j$ is

(a) $2^{2n-1} - \frac{2n!}{n!\,n!}$ (b) $2^{2n-1} - \frac{2.\,2n!}{n!\,n!}$

(c) $2^{2n+1} - \frac{2n!}{2.\,n!}$ (d) $2^{2n-1} - \frac{{}^{2n}C_n}{2}$

55. The value of $99^{50} - 99 \cdot 98^{50} + \frac{99.98}{1.2}(97)^{50} + \ldots + 99$ is

(a) –1 (b) –2

(c) –3 (d) 0

56. If $x = (\sqrt{3}+1)^n$, then $[x]$ is (where $[x]$ denotes the greatest integer less than or equal to x)

(a) $2k$ where $k \in I$ (b) $2k+1$, where $k \in I$

(c) 4^n (d) 8^n

57. The number of irrational terms in the expansion of $(2^{1/5} + 3^{1/10})^{55}$ is

(a) 47 (b) 56

(c) 50 (d) 48

58. If $\frac{C_0}{1} - \frac{C_1}{3} + \frac{C_2}{5} - \ldots + \frac{(-1)^n C_n}{2n+1}$ $= k\int_0^1 x(1-x^2)^{n-1}\,dx$ then $k =$

(a) $\frac{2^{2n+1}.\,n.\,n!}{(2n+1)!}$ (b) $2^{2n}\frac{n!}{(2n+1)!}$

(c) ${}^{2n+1}C_n$ (d) $\frac{2^{2n+1}.\,n.\,(n!)^2}{(2n+1)!}$

59. The value of the expression ${}^{n+1}C_2 + 2\,[{}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2]$ is

(a) $\Sigma\, n$ (b) $\Sigma\, n^2$

(c) $\Sigma\, n^3$ (d) $\frac{(n+1)}{2}$

60. If $n > 3$, then $\Sigma\,(-1)^r (n-r)(n-r+1)(n-r+2)\, C_r =$.

(a) 4 (b) 3

(c) 0 (d) 1

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer(s). For each question write the letters a, b, c, d corresponding to the correct answer(s).

61. The coefficient of x^n in the polynomial $(x + {}^nC_0)(x + 3\,{}^nC_1)(x + 5\,{}^nC_2)\ldots(x + (2n+1)\,{}^nC_n)$ is

(a) $n\,.\,2^n$ (b) $n\,.\,2^{n+1}$

(c) $(n+1)\,.\,2^n$ (d) $n\,.\,2^n + 1$

62. The coefficient of $\lambda^n \mu^n$ in the expansion of $[(1+\lambda)(1+\mu)(\lambda+\mu)]^n$ is

(a) $\sum_{r=0}^{n} C_r^2$ (b) $\sum_{r=0}^{n} C_{r+2}^2$

(c) $\sum_{r=0}^{n} C_{r+3}^2$ (d) $\sum_{r=0}^{n} C_r^3$

63. If $(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n}x^{2n}$ then

(a) $a_0 - a_2 + a_4 - a_6 + \ldots. = 0$, if n is odd.

(b) $a_1 - a_3 + a_5 - a_7 + \ldots. = 0$, if n is even.

(c) $a_0 - a_2 + a_4 - a_6 + \ldots. = 0$, if $n = 4p$, $p \in I^+$

(d) $a_1 - a_3 + a_5 - a_7 + \ldots. = 0$, if $n = 4p+1$, $p \in I^+$

64. The value of $C_0^2 + 3C_1^2 + 5C_2^2 + \ldots$ to $(n+1)$ terms, is (given that $C_r = {}^nC_r$)

(a) ${}^{2n-1}C_{n-1}$

(b) $(2n+1) \cdot {}^{2n-1}C_n$

(c) $2(n+1) \cdot {}^{2n-1}C_n$

(d) ${}^{2n-1}C_n + (2n+1) \cdot {}^{2n-1}C_{n-1}$

65. If n is even positive integer, then the condition that the greatest term in the expansion of $(1+x)^n$ may have the greatest coefficient also is

(a) $\frac{n}{n+2} < x < \frac{n+2}{n}$

(b) $\frac{n+1}{n} < x < \frac{n}{n+1}$

(c) $\frac{n}{n+4} < x < \frac{n+4}{n}$

(d) None of these

66. The number of distinct terms in the expansion of $(x + 2y - 3z + 5w - 7u)^n$ is

(a) $n + 1$

(b) ${}^{n+4}C_4$

(c) ${}^{n+4}C_n$

(d) $\frac{(n+1)(n+2)(n+3)(n+4)}{24}$

67. If n is a positive integer and $(3\sqrt{3}+5)^{2n+1} = \alpha + \beta$ where α is an integer and $0 < \beta < 1$ then

(a) α is an even integer

(b) $(\alpha + \beta)^2$ is divisible by 2^{2n+1}

(c) The integer just below $(3\sqrt{3}+5)^{2n+1}$ divisible by 3

(d) α is divisible by 10

68. For $1 < r < n$, the value of ${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \ldots\ldots + {}^rC_r$ is

(a) ${}^nC_{r+1}$ (b) ${}^{n+1}C_r$

(c) ${}^{n+1}C_{r+1}$ (d) None of these

69. If $(8 + 3\sqrt{7})^n = P + F$, where P is an integer and F is a proper fraction then

(a) P is an odd integer

(b) P is an even integer

(c) $F.(P+F) = 1$

(d) $(1-F)(P+F) = 1$

70. The remainder of 7^{103} when divided by 25 is

(a) 7 (b) 25

(c) 18 (d) 9

71. In the expansion of $\left(\sqrt[3]{4} + \frac{1}{\sqrt[4]{6}}\right)^{20}$,

(a) The number of rational terms = 4

(b) The number of irrational terms = 19

(c) The middle term is irrational

(d) the number of irrational terms = 17

72. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4}$ is equal to

(a) $\frac{a_2}{a_2+a_3}$ (b) $\frac{1}{2} \cdot \frac{a_2}{a_2+a_3}$

(c) $\frac{2a_2}{a_2+a_3}$ (d) $\frac{2a_3}{a_2+a_3}$

73. The coefficient of x^r $(0 \le r \le (n-1))$ in the expansion of $(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \ldots + (x+2)^{n-1}$ is

(a) ${}^nC_r(3^r - 2^n)$ (b) ${}^nC_r(3^{n-r} - 2^{n-r})$

(c) ${}^nC_r(3^r + 2^{n-r})$ (d) None of these

74. Let $a_n = \frac{(1000)^n}{n!}$ for $n \in N$. Then a_n is greatest when

(a) $n = 998$ (b) $n = 999$

(c) $n = 1000$ (d) $n = 1001$

75. If there is a term containing x^{2r} in $\left(x + \frac{1}{x^2}\right)^{n-3}$, then

(a) $n - 2r$ is a positive integral multiple of 3

(b) $n - 2r$ is even

(c) $n - 2r$ is odd

(d) None of these

76. The value of the sum of the series $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + \ldots$ upto $(n+1)$ terms is

(a) 0 (b) 3^n

(c) 5^n (d) None of these

77. If $C_0, C_1, C_2, ..., C_n$ are coefficients in the binomial expansion of $(1+x)^n$, then $C_0C_2 + C_1C_3 + C_2C_4 + \ldots + C_{n-2}C_n$ is equal to

(a) $\frac{2n!}{(n-2)!\,(n+2)!}$

(b) $\frac{2n!}{((n-2)!)^2}$

(c) $\frac{2n!}{((n+2)!)^2}$

(d) ${}^{2n}C_{n-2}$

78. If $a+b=1$ then $\sum_{r=0}^{n} r\,{}^nC_r\,a^r b^{n-r}$ equals

(a) 1 (b) n
(c) na (d) nb

79. If n is a positive integer and $C_k = {}^nC_k$, then $\sum_{k=1}^{n} k^3 \left(\frac{C_k}{C_{k-1}}\right)^2$ equals

(a) $\frac{n(n+1)(n+2)}{12}$

(b) $\frac{n(n+1)^2(n+2)}{12}$

(c) $\frac{n(n+1)(n+2)^2}{12}$

(d) $\frac{n^2(n+1)^2(n+2)^2}{144}$

80. The coefficient of x^{50} in the expansion of $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots + \ldots + 1001\,x^{1000}$ is

(a) ${}^{1000}C_{50}$ (b) ${}^{1001}C_{50}$
(c) ${}^{1002}C_{50}$ (d) ${}^{1000}C_{51}$

81. If $(6\sqrt{6}+14)^{n+1} = m$ and if f is the fractional part of m, then fm is equal to

(a) 15^{n+1} (b) 20^{n+1}
(c) 25^n (d) None of these

82. The number of terms in the expansion of $[(a+3b)^2(3a-b)^2]^3$ is

(a) 14 (b) 28
(c) 32 (d) 56

83. If the sum of the coefficients in the expansion of $(x-2y+3z)^n = 128$, then the greatest coefficient in the expansion of $(1+x)^n$ is

(a) 35 (b) 20
(c) 10 (d) 5

84. The expression $(x+\sqrt{(x^3-1)})^5 + (x-\sqrt{x^3-1})^5$ is a polynomial of degree

(a) 15 (b) 6
(c) 7 (d) 8

85. In the expansion of $(x+y+z)^{25}$

(a) every term is of the form ${}^{25}C_r.\,{}^rC_k.\,x^{25-r}.\,y^{r-k}.\,z^k$

(b) the coefficient of $x^8y^9z^9$ is 0

(c) the number of terms is 351

(d) none of these

86. If $n>3$ and $a, b \in R$, then the value of $ab - n(a-1)(b-1) + \frac{n(n-1)}{1.2}(a-2)(b-2) - \ldots + (-1)^n(a-n)(b-n)$ is

(a) $a^n + b^n$ (b) $\frac{a^n - b^n}{a-b}$
(c) $(ab)^n$ (d) 0

87. The value of $\sum_{i=0}^{n} \sum_{j=1}^{n} {}^nC_j\,{}^jC_i,\ i<j$ is

(a) $3^n - 1$ (b) 0
(c) 2^n (d) none of these

88. The coefficient of $a^{10}b^7c^3$ in the expansion of $(bc+ca+ab)^{10}$ is

(a) 30 (b) 60
(c) 120 (d) 240

89. If $(1+2x+3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, then

(a) $a_1 = 20$ (b) $a_2 = 210$
(c) $a_4 = 8085$ (d) $a_{20} = 2^2.3^7.7$

90. The coefficient of the middle term in the expansion of $(1+x)^{2n}$ is

(a) ${}^{2n}C_n$ (b) $\frac{1.3.5\ldots(2n-1)}{n!}2^n$

(c) 2.6. ..., $(4n-2)$ (d) None of these

Practice Test

MM : 20 **Time : 30 Min.**

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. If $(1+x+2x^2)^{20} = a_0 + a_1x + a_2x^2 + \dots + a_{40}x^{40}$ then $a_1 + a_3 + a_5 + \dots + a_{37}$ equals

(a) $2^{19}(2^{20} - 21)$ (b) $2^{20}(2^{19} - 19)$

(c) $2^{19}(2^{20} + 21)$ (d) None of these

2. If maximum and minimum values of the determinant

$$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

are α and β then

(a) $\alpha^3 - \beta^{17} = 26$

(b) $\alpha + \beta^{97} = 4$

(c) $(\alpha^{2n} - \beta^{2n})$ is always an even integer for $n \in N$.

(d) a triangle can be constructed having its sides as α, β and $\alpha - \beta$

3. The coefficient of x^{50} in $(1+x)^{41}(1-x+x^2)^{40}$ is

(a) 1 (b) 2

(c) 3 (d) 0

4. The number of terms in the expansion of $(\sqrt[4]{9} + \sqrt[6]{8})^{500}$ which are integers is given by

(a) 501 (b) 251

(c) 601 (d) 451

5. If n is an even integer and a, b, c are distinct, the number of distinct terms in the expansion of $(a+b+c)^n + (a+b-c)^n$ is

(a) $\left(\frac{n}{2}\right)^2$ (b) $\left(\frac{n+1}{2}\right)^2$

(c) $\left(\frac{n+2}{2}\right)^2$ (d) $\left(\frac{n+3}{2}\right)^2$

6. Let $a_0, a_1, a_2, \dots$ be the coefficients in the expansion of $(1+x+x^2)^n$ arranged order of x. The value of $a_r - {}^nC_1 a_{r-1} + {}^nC_2 a_{r-2} - \dots + (-1)^r\, {}^nC_r a_0 = \dots$ where r is not divisible by 3.

(a) 5 (b) 3

(c) 1 (d) 0

7. In the expansion of $(3^{-x/4} + 3^{5x/4})^n$ the sum of the binomial coefficients is 64 and the term with the greatest binomial coefficients exceeds the third by $(n-1)$ the value of x must be

(a) 0 (b) 1

(c) 2 (d) 3

8. The last digit of $3^{3^{4n}} + 1$ is

(a) 1 (b) 2

(c) 3 (d) 4

9. If $\frac{1}{1!\,11!} + \frac{1}{3!\,9!} + \frac{1}{5!\,7!} = \frac{2^n}{m!}$ and $f(x+y) = f(x).f(y)\ \forall\ x, y$, $f(1) = 1$, $f'(0) = 10$ then

(a) $f'(n) = m$ (b) $f'(m) = n$

(c) $f'(n) \neq f'(m)$ (d) None of these

10. The number $101^{100} - 1$ is divisible by

(a) 100 (b) 1000

(c) 10000 (d) 100000

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (b)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (c)
7. (a)	**8.** (c)	**9.** (d)	**10.** (c)	**11.** (a)	**12.** (b)
13. (d)	**14.** (d)	**15.** (c)	**16.** (a)	**17.** (d)	**18.** (b)
19. (c)	**20.** (c)	**21.** (b)	**22.** (d)	**23.** (b)	**24.** (d)
25. (b)	**26.** (a), (b)	**27.** (d)	**28.** (c)	**29.** (c)	**30.** (b)
31. (b)	**32.** (c)	**33.** (b)	**34.** (c)	**35.** (a)	**36.** (c)
37. (a), (c)	**38.** (b)	**39.** (b)	**40.** (c)	**41.** (a)	**42.** (c)
43. (a)	**44.** (c)	**45.** (a)	**46.** (b)	**47.** (c)	**48.** (a)
49. (d)	**50.** (b)	**51.** (c)	**52.** (c)	**53.** (d)	**54.** (d)
55. (d)	**56.** (a)	**57.** (c)	**58.** (d)	**59.** (b)	**60.** (c)

Multiple Choice –II

61. (c)	**62.** (d)	**63.** (a), (b)	**64.** (c), (d)	**65.** (a)	**66.** (b), (c), (d)
67. (a), (b), (d)	**68.** (c)	**69.** (a), (d)	**70.** (c)	**71.** (b), (c)	**72.** (c)
73. (b)	**74.** (b), (c)	**75.** (a)	**76.** (a)	**77.** (a), (d)	**78.** (c)
79. (b)	**80.** (c)	**81.** (b)	**82.** (b)	**83.** (a)	**84.** (b), (c)
85. (a), (b), (c)	**86.** (d)	**87.** (a)	**88.** (c)	**89.** (a), (b), (c)	**90.** (a), (b)

Practice Test

1. (a)	**2.** (a), (b), (c)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (d)
7. (a)	**8.** (a), (d)	**9.** (b)	**10.** (a), (b), (c)		

6

DETERMINANTS

§ 6.1. Definition

Determinant of order 2, 3 and 4 are written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

where $a_{ij} \in C \; \forall \; i, j$

§ 6.2. Minors and Cofactors

If we delete the row and column passing through the element a_{ij}, thus obtained is called the minor of a_{ij} and is usually denoted by M_{ij} and cofactors of a_{ij} is $(-1)^{i+j} M_{ij}$ and it is denoted by A_{ij} or C_{ij}.

Let
$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then
$$\Delta = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$
$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}.$$

§ 6.3. Properties of Determinants

(i) The determinant remains unaltered if its rows and columns are interchanged.

(ii) The interchange of any two rows (columns) in Δ changes its sign

(iii) If all the elements of a row (column) in Δ are zero or if two rows (columns) are identical (or proportional), then the value of Δ is zero.

(iv) If all the elements of one row (or column) is multiplied by a non zero number k, then the value of the new determinant is k times the value of the original determinant.

(v) If the elements of a row (column) of a determinant are multiplied by a non zero number k and then added to the corresponding elements of another row (column), then the value of the determinant remains unaltered.

(vi) If Δ becomes zero on putting $x = \alpha$, then we say that $(x - \alpha)$ is a factor of Δ.

(vii)
$$\begin{vmatrix} a_1 + \lambda_1 & b_1 & c_1 \\ a_2 + \lambda_2 & b_2 & c_2 \\ a_3 + \lambda_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & b_1 & c_1 \\ \lambda_2 & b_2 & c_2 \\ \lambda_3 & b_3 & c_3 \end{vmatrix}$$

(viii)
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3 = \begin{vmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(ix)
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Note that we can also multiply rows by columns or columns by rows or columns by columns.

§ 6.4. Systems of Linear Equations

The system of homogeneous linear equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

and $$a_3x + b_3y + c_3z = 0$$

has a non trivial solution (*i.e.* at least one of x, y, z is non zero) if

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

and if $\Delta \neq 0$, then $x = y = z = 0$ is the only solution of above system (Trivial solution).

Cramer's Rule : Let us consider a system of equations

$$a_1x + b_1y + c_1z = d_1 ;$$
$$a_2x + b_2y + c_2z = d_2 ;$$
$$a_3x + b_3y + c_3z = d_3 ;$$

Here $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

By Cramer's rule, we have.

$$x = \frac{\Delta_1}{\Delta}, \; y = \frac{\Delta_2}{\Delta} \text{ and } z = \frac{\Delta_3}{\Delta}.$$

Remarks :

(i) $\Delta \neq 0$, then system will have unique finite solution, and so equations are consistent.

(ii) If $\Delta = 0$, and at least one of $\Delta_1, \Delta_2, \Delta_3$ be non zero, then the system has no solution *i.e.*, equations are inconsistent.

(iii) If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ then equations will have infinite number of solutions, and at least one cofactor of Δ is non zero, *i.e.* equations are consistent.

§ 6.5.. Differentiation of Determinant Function

If $$F(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$$

where $f_1, f_2, f_3; g_1, g_2, g_3; h_1, h_2, h_3$; are the functions of x, then

$$\therefore \quad F'(x) = \begin{vmatrix} f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1' & h_2' & h_3' \end{vmatrix}$$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then

(a) $abc > 1$ (b) $abc > -8$
(c) $abc < -8$ (d) $abc > -2$.

2. Given that $q^2 - pr < 0$, $p > 0$ the value of $\begin{vmatrix} p & q & px+qy \\ q & r & qx+ry \\ px+qy & qx+ry & 0 \end{vmatrix}$ is

(a) zero (b) positive
(c) negative (d) $q^2 + pr$

3. If $f(n) = \alpha^n + \beta^n$ and
$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$
$= k(1-\alpha)^2(1-\beta)^2(1-\gamma)^2$, then $k =$

(a) 1 (b) – 1
(c) $\alpha\beta$ (d) $\alpha\beta\gamma$

4. If x, y, z are integers in A.P., lying between 1 and 9, and $x\,51$, $y\,41$ and $z\,31$ are three-digits numbers then the value of $\begin{vmatrix} 5 & 4 & 3 \\ x\,51 & y\,41 & z\,31 \\ x & y & z \end{vmatrix}$ is

(a) $x + y + z$ (b) $x - y + z$
(c) 0 (d) $x + 2y + z$

5. If
$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}, \ \Delta_2 = \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix}$$ then

(a) $\Delta_1 + \Delta_2 = 0$ (b) $\Delta_1 + 2\Delta_2 = 0$
(c) $\Delta_1 = \Delta_2$ (d) $\Delta_1 = 2\Delta_2$

6. If α, β are non real numbers satisfying $x^3 - 1 = 0$ then the value of $\begin{vmatrix} \lambda+1 & \alpha & \beta \\ \alpha & \lambda+\beta & 1 \\ \beta & 1 & \lambda+\alpha \end{vmatrix}$ is equal to

(a) 0 (b) λ^3
(c) $\lambda^3 + 1$ (d) $\lambda^3 - 1$

7. If $z = \begin{vmatrix} 3+2i & 5-i & 7-3i \\ i & 2i & -3i \\ 3-2i & 5+i & 7+3i \end{vmatrix}$, then

(a) z is purely real
(b) z is purely imaginary
(c) z is mixed complex number, with imaginary part positive
(d) None of these

8. In a third order determinant a_{ij} denotes the element in the ith row and the jth column

If $a_{ij} = \begin{cases} 0, & i = j \\ 1, & i > j \\ -1, & i < j \end{cases}$

then the value of the determinant

(a) 0 (b) 1
(c) – 1 (d) None of these

9. If
$$\Delta(x) = \begin{vmatrix} x & 1+x^2 & x^3 \\ \log(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix},$$
then

(a) $\Delta(x)$ is divisible by x
(b) $\Delta(x) = 0$
(c) $\Delta'(x) = 0$
(d) None of these

10. The value of the determinant $\begin{vmatrix} \log_a(x/y) & \log_a(y/z) & \log_a(z/x) \\ \log_b(y/z) & \log_b(z/x) & \log_b(x/y) \\ \log_c(z/x) & \log_c(x/y) & \log_c(y/z) \end{vmatrix}$ is

(a) 1 (b) – 1
(c) $\log_a xyz$ (d) None of these

11. If $\sqrt{-1} = i$, and ω is a non real cube root of unity then the value of $\begin{vmatrix} 1 & \omega^2 & 1+i+\omega^2 \\ -i & -1 & -1-i+\omega \\ 1-i & \omega^2-1 & -1 \end{vmatrix}$ is equal to

(a) 1 (b) i
(c) ω (d) 0

12. If the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$ then the constant term in the expansion is

(a) 1 (b) 2
(c) –1 (d) –2

13. Using the factor theorem it is found that $b+c$, $c+a$ and $a+b$ are three factors of the determinant $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$ The other factor in the value of the determinant is

(a) 4 (b) 2
(c) $a+b+c$ (d) None of these

14. The value of the determinant $\begin{vmatrix} 1 & e^{i\pi/3} & e^{i\pi/4} \\ e^{-i\pi/3} & 1 & e^{2i\pi/3} \\ e^{-i\pi/4} & e^{-2i\pi/3} & 1 \end{vmatrix}$ is

(a) $2+\sqrt{2}$ (b) $-(2+\sqrt{2})$
(c) $-2+\sqrt{3}$ (d) $2-\sqrt{3}$

15. If
$$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}$$
then $f(200)$ is equal to

(a) 1 (b) 0
(c) 200 (d) –200

16. If
$$D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$$
and $\sum_{k=1}^{n} D_k = 56$ then n equals

(a) 4 (b) 6
(c) 8 (d) None of these

17. If a, b, c are sides of a triangle and
$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0, \text{ then}$$

(a) ΔABC is an equilateral triangle
(b) ΔABC is a right angled isosceles triangle
(c) ΔABC is an isosceles triangle
(d) None of these

18. If the system of equations $2x-y+z=0$, $x-2y+z=0$, $tx-y+2z=0$ has infinitely many solutions and $f(x)$ be a continuous function, such that $f(5+x)+f(x)=2$ then
$$\int_0^{-2t} f(x)\,dx =$$

(a) 0 (b) $-2t$
(c) 5 (d) t

19. If $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x)$ then $f(x)$ is equal to

(a) $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(b) $(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$
(c) $2(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(d) None of these

20. For positive numbers x, y, z the numerical value of the determinant
$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 3 & \log_y z \\ \log_z x & \log_z y & 5 \end{vmatrix} \text{ is}$$

(a) 0
(b) $\log x \log y \log z$
(c) 1
(d) 8

21. If $f(x) = ax^2 + bx + c$, $a, b, c \in R$ and equation $f(x) - x = 0$ has imaginary roots α and β and γ, δ be the roots of $f(f(x)) - x = 0$ then $\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$ is

(a) 0
(b) purely real
(c) purely imaginary
(d) none of these

22. If n is a positive integer, then
$$\begin{vmatrix} {}^{n+2}C_n & {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} \\ {}^{n+3}C_{n+1} & {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} \\ {}^{n+4}C_{n+2} & {}^{n+5}C_{n+3} & {}^{n+6}C_{n+4} \end{vmatrix} =$$

(a) 3 (b) –1
(c) –5 (d) –9

23. Let $\Delta(x) = \begin{vmatrix} (x-2) & (x-1)^2 & x^3 \\ (x-1) & x^2 & (x+1)^3 \\ x & (x+1)^2 & (x+2)^3 \end{vmatrix}$

Then the coefficient of x in $\Delta(x)$ is

(a) –3 (b) –2
(c) –1 (d) 0

24. If $f(x) = \begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix}$

then the value of $\int_{-\pi/2}^{\pi/2} f(x)\,dx$ is equal to

(a) 5 (b) 3
(c) 1 (d) 0

25. If $\sin 2x = 1$, then

$\begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$ equals

(a) 3 (b) 2
(c) 1 (d) None of these

26. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 4 (b) 2
(c) 0 (d) –2

27. If $a, b, c > 0$ and $x, y, z \in R$, then the value of the determinant

$\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^y + b^{-y})^2 & (b^y - b^{-y})^2 & 1 \\ (c^z + c^{-z})^2 & (c^z - c^{-z})^2 & 1 \end{vmatrix}$ is

(a) 6 (b) 4
(c) 2 (d) 0

28. If $A + B + C = \pi$, then

$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$ is equal to

(a) 1 (b) 0
(c) –1 (d) 2

29. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$

Then $f'(\pi/2) =$

(a) 8 (b) 6
(c) 4 (d) 2

30. The value of

$\begin{vmatrix} (-1)^n a & (-1)^{n+1} b & (-1)^{n+1} c \\ a+1 & 1-b & 1+c \\ a-1 & b+1 & 1-c \end{vmatrix} + \begin{vmatrix} (-1)^{n+1} a & a+1 & a-1 \\ (-1)^n b & 1-b & b+1 \\ (-1)^{n+2} c & 1+c & 1-c \end{vmatrix}$ is equal to

(a) 3 (b) 1
(c) –1 (d) None of these

31. If 3^n is a factor of the determinant

$\begin{vmatrix} 1 & 1 & 1 \\ {}^nC_1 & {}^{n+3}C_1 & {}^{n+6}C_1 \\ {}^nC_2 & {}^{n+3}C_2 & {}^{n+6}C_2 \end{vmatrix}$

then the maximum value of n is

(a) 7 (b) 5
(c) 3 (d) 1

32. If a, b, c are non zero real numbers and if the equations $(a-1)x = y+z$, $(b-1)y = z+x$, $(c-1)z = x+y$ has a non trivial solution then $ab + bc + ca$ equals

(a) $a + b + c$ (b) abc
(c) 1 (d) None of these

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

33. The determinant

$$\Delta = \begin{vmatrix} a^2 + x^2 & ab & ac \\ ab & b^2 + x^2 & bc \\ ac & bc & c^2 + x^2 \end{vmatrix}$$

is divisible by

(a) x (b) x^2
(c) x^3 (d) x^4

34. If

$$D_k = \begin{vmatrix} 2^{k-1} & \dfrac{1}{k(k+1)} & \sin k\theta \\ x & y & z \\ 2^n - 1 & \dfrac{n}{n+1} & \dfrac{\sin\left(\dfrac{n+1}{2}\right)\theta \sin \dfrac{n}{2}\theta}{\sin \theta/2} \end{vmatrix}$$

then $\sum_{k=1}^{n} D_k$ is equal to

(a) 0
(b) independent of n
(c) independent of 0
(d) independent of x, y and z

35. The value of the determinant $\begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}\,i & 3\sqrt{2}+\sqrt{6}\,i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}\,i & \sqrt{27}+2i \end{vmatrix}$ is

(a) Complex (b) Real
(c) Irrationa (d) Rational

36. The determinant

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$$

is equal to zero, if

(a) a, b, c are in A.P.
(b) a, b, c are in G.P.
(c) a, b, c are in H.P.
(d) α is a root of $ax^2 + bx + c = 0$
(e) $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

37. The digits A, B, C are such that the three digit numbers $A88$, $6B8$, $86C$ are divisible by 72 then the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by

(a) 72 (b) 144
(c) 288 (d) 216

38. If $a > b > c$ and the system of equations $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$ has a non trivial solution then both the roots of the quadratic equation $at^2 + bt + c = 0$ are

(a) real (b) of opposite sign
(c) positive (d) complex

39. The roots of the equation

$$\begin{vmatrix} 3x^2 & x^2 + x\cos\theta + \cos^2\theta & x^2 + x\sin\theta + \sin^2\theta \\ x^2 + x\cos\theta + \cos^2\theta & 3\cos^2\theta & 1 + \dfrac{\sin 2\theta}{2} \\ x^2 + x\sin\theta + \sin^2\theta & 1 + \dfrac{\sin 2\theta}{2} & 3\sin^2\theta \end{vmatrix} = 0$$

are

(a) $\sin\theta$, $\cos\theta$ (b) $\sin^2\theta$, $\cos^2\theta$
(c) $\sin\theta$, $\cos^2\theta$ (d) $\sin^2\theta$, $\cos\theta$

40. In a triangle ABC the value of the determinant

$$\begin{vmatrix} \sin\dfrac{A}{2} & \sin\dfrac{B}{2} & \sin\dfrac{C}{2} \\ \sin(A+B+C) & \sin\dfrac{B}{2} & \cos\dfrac{A}{2} \\ \cos\left(\dfrac{A+B+C}{2}\right) & \tan(A+B+C) & \sin\dfrac{C}{2} \end{vmatrix}$$

is less than or equal to

(a) 1/2 (b) 1/4
(c) 1/8 (d) None of these

41. If $At^4 + Bt^3 + Ct^2 + Dt + E = \begin{vmatrix} t^2 + 3t & t-1 & t-3 \\ t+1 & 2-t & t-3 \\ t-3 & t+4 & 3t \end{vmatrix}$

then E equals

(a) 33 (b) −39
(c) 27 (d) 24

42. If α, β, γ are real numbers, then

$$\Delta = \begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos(\gamma - \alpha) \\ \cos(\alpha - \beta) & 1 & \cos(\gamma - \beta) \\ \cos(\alpha - \gamma) & \cos(\beta - \gamma) & 1 \end{vmatrix}$$

is equal to

(a) −1 (b) $\cos\alpha \cos\beta \cos\gamma$
(c) $\cos\alpha + \cos\beta + \cos\gamma$ (d) None of these

43. If all elements of a third order determinant are equal to 1 or −1, then the determinant itself is

(a) an odd number (b) an even number
(c) an imaginary number (d) a real number

44. The coefficient of x in the determinant $\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$ is

(a) 4 (b) 2
(c) 0 (d) −2

45. If α, β be the roots of the equation $ax^2 + bx + c = 0$ Let $S_n = \alpha^n + \beta^n$ for $n \geq 1$

Let $\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$

then $\Delta =$

(a) $\dfrac{s^2(b^2-4ac)}{a^4}$ (b) $\dfrac{(a+b+c)^2(b^2-4ac)}{a^4}$

(c) $\dfrac{b^2-4ac}{a^4}$ (d) $\dfrac{(a+b+c)^2}{4}$

Practice Test

M.M : 20 Time : 30 Min

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. If $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix}$

$= (x-y)(y-z)(z-x)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$ then

(a) $k = -2$ (b) $k = -1$

(c) $k = 0$ (c) $k = 1$

2. If $f_r(x), g_r(x), h_r(x), r = 1, 2, 3$ are polynomials in x such that $f_r(a) - g_r(a) - h_r(a), r = 1, 2, 3$ and

$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

then $F'(x)$ at $x = a$ is

(a) 1 (b) 2

(c) 3 (d) None of these

3. The largest value of a third order determinant whose elements are equal to 1 or 0 is

(a) 0 (b) 2

(c) 4 (d) 6

4. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then $\dfrac{d^3}{dx^3}\{f(x)\}$ at $x = 0$ is

(a) p (b) $p + p^2$

(c) $p + p^3$ (d) independent of p

5. Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \lambda$

then the value of $5\alpha + 4\beta + 3\gamma + 2\delta + \lambda =$

(a) –11 (b) 0

(c) –16 (d) 16

6. Let

$\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$; $0 \leq \theta < 2\pi$ then

(a) $\Delta = 0$ (b) $\Delta \in (0, \infty)$

(c) $\Delta \in [-1, 2]$ (d) $\Delta \in [2, 4]$

7. Let $f(x) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$,

where the symbols have their usual meanings. The $f(x)$ is divisible by

(a) $n^2 + n + 1$ (b) $(n+1)!$

(c) $n!$ (d) none of these

8. Eliminating a, b, c, from

$x = \dfrac{a}{b-c}, y - \dfrac{b}{c-a}, z = \dfrac{c}{a-b}$ we get

(a) $\begin{vmatrix} 1 & -x & x \\ 1 & -y & y \\ 1 & -z & z \end{vmatrix} = 0$

(b) $\begin{vmatrix} 1 & -x & x \\ 1 & 1 & -y \\ 1 & z & 1 \end{vmatrix} = 0$

(c) $\begin{vmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{vmatrix} = 0$

(d) None of these

9. The system of equations

$x - y\cos\theta + z\cos 2\theta = 0,$

$-x\cos\theta + y - z\cos\theta = 0,$

$x\cos 2\theta - y\cos\theta + z = 0,$

has non trivial solution for θ equals to

(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) $\pi/12$

10. Let $\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$ and $\int_0^2 \Delta(x)\,dx = -16$ where a, b, c, d are in A.P., then the common difference of the A.P. is equal to

(a) $+1$ (b) ± 2
(c) ± 3 (d) ± 4

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (a)	**6.** (b)
7. (a)	**8.** (a)	**9.** (a)	**10.** (d)	**11.** (d)	**12.** (c)
13. (a)	**14.** (b)	**15.** (b)	**16.** (d)	**17.** (c)	**18.** (b)
19. (a)	**20.** (d)	**21.** (b)	**22.** (b)	**23.** (b)	**24.** (d)
25. (d)	**26.** (c)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (d)
31. (c)	**32.** (b)				

Multiple Choice –II

33. (a), (b), (c), (d)		**34.** (a), (b), (c), (d)		**35.** (a), (b)	**36.** (b), (e)
37. (a), (b), (c)	**38.** (a), (b)	**39.** (a)	**40.** (c)	**41.** (b)	**42.** (d)
43. (b)	**44.** (c)	**45.** (b)			

Practice Test

1. (b)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (d)
7. (a), (c)	**8.** (b), (c)	**9.** (a), (b), (c), (d)		**10.** (b)	

7

PROBABILITY

§ 7.1. Definition

The probability of an event to occur is the ratio of the number of cases in its favour to the total number of cases (equally likely).

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of favourable cases}}{\text{total number of cases}}$$

Remark : If a is the number of cases favourable to the event E, b is the number of cases favourable to the event E', then *odds in favour* of E are $a : b$ and *odds against of* E are $b : a$

In this case $$P(E) = \frac{a}{a+b}$$

and $$P(E') = \frac{b}{a+b}$$

$$\therefore \quad P(E) + P(E') = 1.$$

$0 < P(E) < 1$ therefore maximum value of $P(E) = 1$ and the minimum value of $P(E) = 0$.

§ 7.2. Type of Events :

(i) **Equally likely Events :** The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

(ii) **Independent Events :** Two events are said to be independent if the occurrence of one does not depend upon the other. If a set of events $E_1, E_2, ..., E_n$ for Independent Events.

$$P(E_1 \cap E_2 \cap E_3 \cap .... \cap E_n) = P(E_1) \cdot P(E_2) .... P(E_n)$$

(iii) **Mutually Exclusive Events :** A set of events is said to be *mutually exclusive* if occurrence of one of them precludes the occurrence of any of the remaining events.

If a set of events $E_1, E_2, \ldots, E_n$

for mutually exclusive events. Here $P(E_1 \cap E_2 \cap ..... \cap E_n) = \phi$

then $$P(E_1 \cup E_2 \cup .... \cup E_n) = P(E_1) + P(E_2) + P(E_3) + ... + P(E_n)$$

(iv) **Exhaustive Events :** A set of events is said to be *Exhaustive* if the performance of the experiment results in the occurrence of at least one of them

If a set of Events $E_1, E_2, ..., E_n$ then for Exhaustive Events

$$P(E_1 \cup E_2 \cup .... \cup E_n) = 1$$

(v) **Mutually Exclusive and Exhaustive Events :** A set of events is said to be mutually exclusive and exhaustive if above two conditions are satisfied.

If a set of Events $E_1, E_2, ..., E_n$ then for mutually exclusive and exhaustive events

$$P(E_1 \cup E_2 \cup .... \cup E_n) = P(E_1) + P(E_2) + ... + P(E_n) = 1$$

(vi) **Compound Events :** If $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive events then, if E is any event

$$\therefore \quad P(E) = \sum_{i=1}^{n} P(E \cap E_i) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right) \text{ if } P(E_i) > 0$$

§ 7.3. Conditional Probability

The probability of occurrence of an event E_1, given that E_2 has already occured is called the conditional probability of occurrence of E_1 on the condition that E_2 has already occured. It is denoted by $P\left(\frac{E_1}{E_2}\right)$

i.e.. $$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)},\ E_2 \neq \phi$$

§ 7.4. Baye's Theorem or Inverse Probability :

If $E_1, E_2, \dots, E_n$ are n mutually exclusive and exhaustive events such that

$$P(E_i) > 0 \quad (0 < i < n)$$

and E is any event, then for $1 < k < n$,

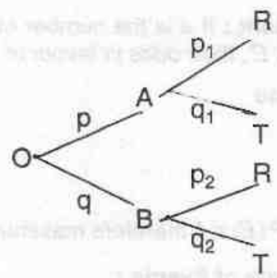
$$P\left(\frac{E_k}{E}\right) = \frac{P(E_k)\, P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n} P(E_k)\, P\left(\frac{E}{E_k}\right)}$$

Remark : We can visulise a tree structure here

$P(A) = p;\ P(B) = q;$

$P\left(\frac{R}{A}\right) = p_1;\ P\left(\frac{T}{A}\right) = q_1;$

$P\left(\frac{R}{B}\right) = p_2;\ P\left(\frac{T}{B}\right) = q_2;$



If we are to find

$P\left(\frac{A}{R}\right)$ we go

$$P\left(\frac{A}{R}\right) = \frac{P\left(\frac{R}{A}\right) P(A)}{P\left(\frac{R}{A}\right) P(A) + P\left(\frac{R}{B}\right) P(B)}$$

§ 7.5. Multinomial Theorem

If a die has m faces marked 1, 2, 3, ..., m and if such n dices are thrown. Then the probability that the sum of the numbers on the upper faces is equal to r is given by

$$\text{the coefficient of } x^r \text{ in } \frac{(x + x^2 + \dots + x^m)^n}{m^n}$$

§ 7.6. Binomial Distribution

Suppose a binomial experiment has probability of success p and that of failure $q\,(p + q = 1)$, then probability of r success in a series of n independent trials is given by

$$P(r) = {}^nC_r\, p^r\, q^{n-r} \text{ where } p + q = 1 \text{ and } r = 0, 1, 2, 3, \dots, n$$

Remarks :

(i) The probability of getting at least k success is

$$P(r > k) = \sum_{r=1}^{n} {}^nC_r\, p^r\, q^{n-r}$$

(ii) The probability of getting at most k success is

$$P(0 < r < k) = \sum_{r=0}^{k} {}^nC_r \cdot p^r \cdot q^{n-r}$$

(iii) The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

Notation : If E_1 and E_2 are two events, then

(i) $E_1 \cup E_2$ stands for occurrence of at least one of E_1, E_2

(ii) $E_1 \cap E_2$ stands for the simultaneous occurrence of E_1, E_2.

(iii) E' or $\bar{E}$ or E^c stands for non occurrence.

(iv) $E_1' \cap E_2'$ stands for non occurrence of both E_1 and E_2.

§ 7.7. Expectation

If p be the probability of success of a person in any venture and M be the sum of money which he will receive in case of success, the sum of money denoted by pM is called expectation.

§ 7.8. Important Results

(i) If E_1 and E_2 are arbitrary events, then

(a) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

(b) P (Exactly one of E_1, E_2 occurs) $= P(E_1 \cap E_2') + P(E_1' \cap E_2)$

$= P(E_1) - P(E_1 \cap E_2) + P(E_2) - P(E_1 \cap E_2)$

$= P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$

$= P(E_1 \cup E_2) - P(E_1 \cap E_2)$

(c) P (Neither E_1 nor E_2) $= P(E_1' \cap E_2') = 1 - P(E_1 \cup E_2)$

(d) $P(E_1' \cup E_2') = 1 - P(E_1 \cap E_2)$

(ii) If E_1, E_2, E_3 are three events then

(a) $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$

(b) P (At least two of E_1, E_2, E_3 occur)

$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 2P(E_1 \cap E_2 \cap E_3)$

(c) P (Exactly two of E_1, E_2, E_3 occur)

$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 3P(E_1 \cap E_2 \cap E_3)$.

(d) P (Exactly one of E_1, E_2, E_3 occur)

$= P(E_1) + P(E_2) + P(E_3) - 2P(E_1 \cap E_2) - 2P(E_2 \cap E_3) - 2P(E_3 \cap E_1) + 3P(E_1 \cap E_2 \cap E_3)$.

(iii) If $E_1, E_2, E_3, \ldots, E_n$ are n events then

(a) $P(E_1 \cup E_2 \cup \ldots \cup E_n) \le P(E_1) + P(E_2) + \ldots + P(E_n)$

(b) $P(E_1 \cap E_2 \cap \ldots \cap E_n) \ge 1 - P(\bar{E}_1) - P(\bar{E}_2) - \ldots - P(\bar{E}_n)$.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. For n independent events A_i's, $P(A_i) = 1/(1+i)$, $i = 1, 2, \ldots, n$. The probability that atleast one of the events occurs is

(a) $1/n$ (b) $1/(n+1)$
(c) $n/(n+1)$ (d) none of these

2. The probabilities that a student will obtain grades A, B, C or D are 0.30, 0.35, 0.20 and 0.15 respectively. The probability that he will receive *atleast* C grade, is

(a) 0.65 (b) 0.85
(c) 0.80 (d) 0.20

3. The probabilitity that a teacher will give an unanounced test during any class meeting is 1/5. If a student is absent twice, the probability that he will miss at least one test, is :

(a) 7/25 (b) 9/25
(c) 16/25 (d) 24/25

4. If the probability for A to fail in an examination is 0.2 and that for B is 0.3 then the probability that either A or B fails, is

(a) 0.38 (b) 0.44
(c) 0.50 (d) 0.94

5. A box contains 15 transistors, 5 of which are defective. An inspector takes out one transistor at random, examines it for defects, and replaces it. After it has been replaced another inspector does the same thing, and then so does a third inspector. The probability that at least one of the inspectors finds a defective transistor, is equal to

(a) 1/27 (b) 8/27
(c) 19/27 (d) 26/27

6. There are 5 duplicate and 10 original items in an automobile shop and 3 items are brought at random by a customer. The probability that none of the items is duplicate, is

(a) 20/91 (b) 22/91
(c) 24/91 (d) 89/91

7. Three letters are written to three different persons and addresses on the three envelopes are also written. Without looking at the addresses, the letters are kept in these envelopes. The probability that all the letters are not placed into their right envelopes is

(a) 1/2 (b) 1/3
(c) 1/6 (d) 5/6

8. A man is known to speak truth in 75% cases. If he throws an unbiased die and tells his friend that it is a six, then the probability that it is actually a six, is

(a) 1/6 (b) 1/8
(c) 3/4 (d) 3/8

9. A bag contains 7 red and 2 white balls and another bag contains 5 red and 4 white balls. Two balls are drawn, one from each bag. The probability that both the balls are white, is

(a) 2/9 (b) 2/3
(c) 8/81 (d) 35/81

10. A bag contains 5 red, 3 white and 2 black balls. If a ball is picked at random, the probability that it is red, is

(a) 1/5 (b) 1/2
(c) 3/10 (d) 9/10

11. A single letter is selected at random from the word 'PROBABILITY'. The probability that it is a vowel, is

(a) 2/11 (b) 3/11
(c) 4/11 (d) None of these

12. 10 bulbs out of a sample of 100 bulbs manufactured by a company are defective. The probability that 3 out of 4 bulbs, bought by a customer will not be defective, is :

(a) [illegible]$C_3/{}^{100}C_4$ (b) ${}^{90}C_3/{}^{96}C_4$
(c) ${}^{90}C_3/{}^{100}C_4$ (d) $({}^{90}C_3 \times {}^{10}C_1)/{}^{100}C_4$

13. Fifteen coupons are numbered 1, 2, 3, ..., 15 respectively. Seven coupons are selected at random one at a time with replacement. The probability that the largest number appearing on a selected coupon is atmost 9, is

(a) $(1/15)^7$ (b) $(3/5)^7$
(c) $(8/15)^7$ (d) None of these

14. The probability that a man aged x years will die in a year is p. The probability that out of n men $M_1, M_2 ..., M_n$, each aged x, M_1 will die and be the first to die, is :

(a) $1/n^2$ (b) $1-(1-p)^n$
(c) $\frac{1}{n^2[1-(1-p)^n]}$ (d) $\frac{1}{n}[1-(1-p)^n]$

15. n letters are written to n different persons and addreses on the n envelopes are also written. If the letters are placed in the envelopes at random, the probability that atleast one letter is not placed in the right envelope, is

(a) $1-\frac{1}{n}$ (b) $1-\frac{1}{2n}$
(c) $1-\frac{1}{n^2}$ (d) $1-\frac{1}{n!}$

16. Three athletes A, B and C participate in a race. Both A and B have the same probability of winning the race and each is twice as likely to win as C. The probability that B or C wins the race is

(a) 2/3 (b) 3/5
(c) 3/4 (d) 13/25

17. A number is chosen at random from among the first 30 natural numbers. The probability of the number chosen being a prime, is

(a) 1/3 (b) 3/10
(c) 1/30 (d) 11/30

18. Out of 13 applicants for a job, there are 8 men and 5 women. It is desired to select 2 persons for the job. The probability that atleast one of the selected persons will be a woman, is

(a) 5/13 (b) 10/13
(c) 14/39 (d) 15/39

19. Two athletes A and B participate in a race along with other athletes. If the chance of A winning the race is 1/6 and that of B winning the same race is 1/8, then the chance that neither wins the race, is

(a) 1/4 (b) 7/24
(c) 17/24 (d) 35/48

20. Three players A, B, C in this order, cut a pack of cards, and the whole pack is reshuffled after each cut. If the winner is one who first draws a diamond then C's chance of winning is
(a) 9/28 (b) 9/37
(c) 9/64 (d) 27/64

21. 6 girls and 5 boys sit together randomly in a row, the probability that no two boys sit together, is :
(a) $\frac{6!5!}{11!}$ (b) $\frac{6!6!}{11!}$
(c) $\frac{6!7!}{2!11!}$ (d) $\frac{5!7!}{2!11!}$

22. A mapping is selected at random from the set of all mappings of the set $A = \{1, 2 ..., n\}$ into itself. The probability that the mapping selected is bijective, is
(a) $\frac{1}{n!}$ (b) $\frac{1}{n^n}$
(c) $\frac{n!}{2^n}$ (d) $\frac{n!}{n^n}$

23. Three letters are written to three different persons and addresses on the three envelopes are also written. Without looking at the addresses, the letters are kept in these envelopes. The probability that the letters go into the right envelopes, is
(a) 1/3 (b) 1/6
(c) 1/9 (d) 1/27

24. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of the four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5, is
(a) 1/81 (b) 16/81
(c) 65/81 (d) 80/81

25. The probability of guessing correctly atleast 8 out of 10 answers on a true-false examination, is
(a) 7/64 (b) 7/128
(c) 45/1024 (d) 175/1024

26. The probability that the 13th day of a randomly chosen month is a second Saturday, is
(a) 1/7 (b) 1/12
(c) 1/84 (d) 19/84

27. A box contains cards numbered 1 to 100. A card is drawn at random from the box. The probability of drawing a number which is a square, is
(a) 1/5 (b) 2/5
(c) 1/10 (d) None of these

28. An integer is chosen at random from the numbers 1,2, .., 25. The probability that the chosen number is divisible by 3 or 4, is
(a) 2/25 (b) 11/25
(c) 12/25 (d) 14/25

29. Two players A and B throw a die alternately for a prize of Rs. 11/- which is to be won by a player who first throws a six. If A starts the game, their respective expectations are
(a) Rs. 6; Rs. 5 (b) Rs. 7; Rs. 4
(c) Rs. 5.50; Rs. 5.50 (d) Rs. 5.75; Rs. 5.25

30. A three-digit number is selected at random from the set of all three-digit numbers. The probability that the number selected has all the three digits same, is
(a) 1/9 (b) 1/10
(c) 1/50 (d) 1/100

31. In a college 20% students fail in mathematics, 25% in Physics, and 12% in both subjects. A student of this college is selected at random. The probability that this student who has failed in Mathematics would have failed in Physics too, is :
(a) 1/20 (b) 3/25
(c) 12/25 (d) 3/5

32. A purse contains 4 copper and 3 silver coins, and a second purse contains 6 copper and 2 silver coins. A coin is taken out from any purse, the probability that it is a copper coin, is
(a) 3/7 (b) 4/7
(c) 3/4 (d) 37/56

33. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three vertices is equilateral, is
(a) $\frac{1}{2}$ (b) $\frac{1}{5}$
(c) $\frac{1}{10}$ (d) $\frac{1}{20}$

34. The probabilities of two events A and B are 0·3 and 0·6 respectively. The probability that both A and B occur simultaneously is 0·18. Then the probability that neither A nor B occurs is

(a) 0·10 (b) 0·28
(c) 0·42 (d) 0·72

35. One mapping is selected at random from all the mappings of the set $A = \{1, 2, 3, .., n\}$ into itself. The probability that the mapping selected is one to one is given by

(a) $\frac{1}{n^n}$ (b) $\frac{1}{n!}$
(c) $\frac{(n-1)!}{n^{n-1}}$ (d) None of these

36. Two persons each makes a single throw with a pair of dice. The probability that the throws are unequal is given by

(a) $\frac{1}{6^3}$ (b) $\frac{73}{6^3}$
(c) $\frac{51}{6^3}$ (d) None of these

37. If the mean and variance of a binomial variate X are 7/3 and 14/9 respectively. Then probability that X takes value 6 or 7 is equal to

(a) $\frac{1}{729}$ (b) $\frac{5}{729}$
(c) $\frac{7}{729}$ (d) $\frac{13}{729}$

38. A bag contains a white and b black balls. Two players A and B alternately draw a ball from the bag, replacing the ball each time after the draw. A begins the game. If the probability of A winning (that is drawing a white ball) is twice the probability of B winning, then the ratio $a : b$ is equal to

(a) 1 : 2 (b) 2 : 1
(c) 1 : 1 (d) None of these

39. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ..., 99. Suppose X and Y are the sum and product of the digit found on the ticket $P(X = 7/Y = 0)$ is given by

(a) 2/3 (b) 2/19
(c) 1/50 (d) None of these

40. Let X be a set containing n elements. Two subsets A and B of X are chosen at random the probability that $A \cup B = X$ is

(a) ${}^{2n}C_n/2^{2n}$ (b) $1/{}^{2n}C_n$
(c) $1.3.5. ... (2n-1)/2^n\, n!$ (d) $(3/4)^n$

41. A natural number x is chosen at random from the first 100 natural numbers. The probability that $x + \frac{100}{x} > 50$ is

(a) 1/10 (b) 11/50
(c) 1/20 (d) None of these

42. If X and Y are independent binomial variates $B(5, 1/2)$ and $B(7, 1/2)$ then $P(X + Y = 3)$ is

(a) 55/1024 (b) 55/4098
(c) 55/2048 (d) None of these

43. A die is rolled three times, the probability of getting a larger number than the previous number each time is

(a) 15/216 (b) 5/54
(c) 13/216 (d) 1/18

44. A sum of money is rounded off to the nearest rupee; the probability that round off error is at least ten paise is

(a) 19/101 (b) 19/100
(c) 82/101 (d) 81/100

45. Eight coins are tossed at a time, the probability of getting atleast 6 heads up, is

(a) $\frac{7}{64}$ (b) $\frac{57}{64}$
(c) $\frac{37}{256}$ (d) $\frac{229}{256}$

46. 10% bulbs manufactured by a company are defective. The probability that out of a sample of 5 bulbs, none is defective, is

(a) $(1/2)^5$ (b) $(1/10)^5$
(c) $(9/10)$ (d) $(9/10)^5$

47. Of the 25 questions in a unit, a student has worked out only 20. In a sessional test of that unit, two questions were asked by the teacher. The probability that the student can solve both the questions correctly, is

(a) 8/25 (b) 17/25
(c) 9/10 (d) 19/30

48. The probability that at least one of the events A and B occur is 0·6. If A and B occur simultaneously with probability 0·2, then $P(\bar{A}) + P(\bar{B})$, where $\bar{A}$ and $\bar{B}$ are complements of A and B respectively, is equal to
(a) 0·4 (b) 0·8
(c) 1·2 (d) 1·4

49. Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8\}$. An element (a, b) of their cartesian product $A \times B$ is chosen at random. The probability that $a + b = 9$, is
(a) 1/5 (b) 2/5
(c) 3/5 (d) 4/5

50. Dialing a telephone number, a man forgot the last two digits and remembering only that they are different, dialled them at random. The probability of the number being dialed correctly is
(a) 1/2 (b) 1/45
(c) 1/72 (d) 1/90

51. If A and B are any two events, then the probability that exactly one of them occurs, is
(a) $P(A \cap \bar{B}) + P(\bar{A} \cap B)$
(b) $P(A \cup \bar{B}) + P(\bar{A} \cup B)$
(c) $P(A) + P(B) - P(A \cap B)$
(d) $P(A) + P(B) + 2P(A \cap B)$

52. A speaks truth in 60% cases and B speaks truth in 70% cases. The probability that they will say the same thing while describing a single event is
(a) 0·56 (b) 0·54
(c) 0·38 (d) 0·94

53. If the integers λ and μ are chosen at random between 1 to 100 then the probability that a number of the form $7^{\lambda} + 7^{\mu}$ is dirisible by 5 is
(a) 1/4 (b) 1/7
(c) 1/8 (d) 1/49

54. If two events A and B are such that $P(A) > 0$ and $B(B) \neq 1$, then $P(\bar{A}/\bar{B})$ is equal to
(a) $1 - P(A/B)$ (b) $1 - P(\bar{A}/B)$
(c) $\dfrac{1 - P(A \cup B)}{P(\bar{B})}$ (d) $\dfrac{P(A)}{P(\bar{B})}$

55. A three digit number, which is multiple of 11, is chosen at random. Probability that the number so chosen is also a multiple of 9 is equal to
(a) 1/9 (b) 2/9
(c) 1/100 (d) 9/100

56. If $\dfrac{1+4p}{4}, \dfrac{1-p}{4}, \dfrac{1-2p}{2}$ are probabilities of three mutually exclusive events, then
(a) $\frac{1}{3} \le p \le \frac{1}{2}$ (b) $\frac{1}{3} \le p \le \frac{2}{3}$
(c) $\frac{1}{6} \le p \le \frac{1}{2}$ (d) None of these

57. A box contains tickets numbered 1 to 20. 3 tickets are drawn from the box with replacement. The probability that the largest number on the tickets is 7 is
(a) 7/20 (b) $1 - (7/20)^3$
(c) 2/10 (d) None of these

58. Two numbers x and y are chosen at random from the set $\{1, 2, 3, ..., 30\}$. The probability that $x^2 - y^2$ is divisible by 3 is
(a) 3/29 (b) 4/29
(c) 5/29 (d) None of these

59. A fair die is thrown until a score of less than five points is obtained. The probability of obtaining less than three points on the last throw is
(a) 3/4 (b) 4/5
(c) 5/6 (d) 1/2

60. Seven digits from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are written in a random order. The probability that this seven digit number is divisible by 9 is
(a) 2/9 (b) 1/5
(c) 1/3 (c) 1/9

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

61. A fair coin is tossed n times. Let X = the number of times head occurs. If $P(X = 4)$, $P(X = 5)$ and $P(X = 6)$ are in A.P. then the value of n can be

(a) 7 (b) 10
(c) 12 (d) 14

62. A and B are two events, the probability that exactly one of them occurs is given by
(a) $P(A) + P(B) - 2(A \cap B)$
(b) $P(A \cup B) - P(A \cap B)$
(c) $P(A') + P(B') - 2P(A' \cap B')$
(d) $P(A \cap B') + P(A' \cap B)$

63. A wire of length l is cut into three pieces. What is the probability that the three pieces form a triangle ?
(a) 1/2 (b) 1/4
(c) 2/3 (d) None of these

64. Suppose X is a binomial variate $B(5, p)$ and $P(X = 2) = P(X = 3)$, then p is equal to
(a) 1/2 (b) 1/3
(c) 1/4 (d) 1/5

65. A bag contains four tickets marked with 112, 121, 211, 222 one ticket is drawn at random from the bag. Let E_i $\{i = 1, 2, 3\}$ denote the event that ith digit on the ticket is 2. Then
(a) E_1 and E_2 are independent
(b) E_2 and E_3 are indepndent
(c) E_3 and E_1 are independent
(d) E_1, E_2, E_3 are independent

66. A bag contains four tickets numbered 00, 01, 10, 11. Four tickets are chosen at random with replacement, the probability that sum of the numbers on the tickets is 23 is
(a) 3/32 (b) 1/64
(c) 5/256 (d) 7/256

67. A natural number is selected at random from the set $X = \{x/1 < x < 100\}$. The probability that the number satisfies the inequation $x^2 - 13x < 30$, is
(a) 9/50 (b) 3/20
(c) 2/11 (d) None of these

68. Two integers x and y are chosen, without replacement, at random from the set $\{x/0 < x < 10, x \text{ is an integer}\}$ the probability that $|x - y| \le 5$ is
(a) $\frac{87}{121}$ (b) $\frac{89}{121}$
(c) $\frac{91}{121}$ (d) $\frac{101}{121}$

69. The adjoining Fig. gives the road plan of lines connecting two parallel roads AB and A_1B_1. A man walking on the road AB takes a turn at random to reach the road A_1B_1. It is known that he reaches the road A_1B_1 from O by taking a straight line. The chance that he moves on a straight line from the road AB to the A_1B_1 is
(a) 0·25 (b) 0·04
(c) 0·2 (d) None of these

70. If $a \in [-20, 0]$, then the probability that the graph of the function $y = 16x^2 + 8(a + 5)x - 7a - 5$ is strictly above the x-axis is
(a) 1/2 (b) 1/17
(c) 17/20 (d) None of these

71. Two distinct numbers are selected at random from the first twelve natural numbers. The probability that the sum will be divisible by 3 is
(a) 1/3 (b) 23/66
(c) 1/2 (d) None of these

72. If A and B are independent events such that $0 < P(A) < 1$, $0 < P(B) < 1$ then
(a) A, B are mutually exclusive
(b) A and $\overline{B}$ are independent
(c) $\overline{A}$, $\overline{B}$ are independent
(d) $P(A/B) + P(\overline{A}/B) = 1$

73. Given that $x \in [0, 1]$ and $y \in [0, 1]$. Let A be the event of (x, y) satisfying $y^2 \le x$ and B be the event of (x, y) satisfying $x^2 \le y$, then
(a) $P(A \cap B) = \frac{1}{3}$ (b) A, B are exclusive
(c) $P(A) = P(B)$ (d) $P(B) < P(A)$

74. The probability that out of 10 persons, all born in April, at least two have the same birthday is
(a) $\frac{{}^{30}C_{10}}{(30)^{10}}$ (b) $1 - \frac{{}^{30}C_{10}}{30\,!}$
(c) $\frac{(30)^{10} - {}^{30}C_{10}}{(30)^{10}}$ (d) None of these

75. A pair of fair dice is rolled together till a sum of either 5 or 7 is obtained, the probability that 5 comes before 7 is
(a) 0·2 (b) 0·3
(c) 0·4 (d) 0·5

76. A second order determinant is writeen down at random using the numbers 1, −1 as elements. The probability that the value of the determinant is non zero is

(a) 1/2 (b) 3/8
(c) 5/8 (d) 1/3

77. A five digit number is chosen at random. The probability that all the digits are distinct and digits at odd place are odd and digits at even places are even is

(a) 1/25 (b) 1/75
(c) 1/37 (d) 1/74

78. A natural number is selected from 1 to 1000 at random, then the probability that a particular non-zero digit appears at most once is

(a) 3/250 (b) 143/250
(c) 243/250 (d) 7/250

79. If A and B are events at the same experiments with $P(A) = 0{\cdot}2$, $P(B) = 0{\cdot}5$, then maximum value of $P(A' \cap B)$ is

(a) 1/4 (b) 1/2
(c) 1/8 (d) 1/16

80. $x_1, x_2, x_3, \ldots, x_{50}$ are fifty real numbers such that $x_r < x_{r+1}$ for $r = 1, 2, 3, \ldots, 49$. Five numbers out of these are picked up at random. The probability that the five numbers have x_{20} as the middle number is

(a) $\dfrac{{}^{20}C_2 \times {}^{30}C_2}{{}^{50}C_5}$ (b) $\dfrac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$

(c) $\dfrac{{}^{19}C_2 \times {}^{31}C_3}{{}^{50}C_5}$ (d) None of these

81. If A and B are two events such that $P(A) = 1/2$ and $P(B) = 2/3$, then

(a) $P(A \cup B) \geq 2/3$
(b) $P(A \cap B') < 1/3$
(c) $1/6 < P(A \cap B) < 1/2$
(d) $1/6 \leq P(A' \cap B) < 1/2$

82. For two events A and B, if $P(A) = P(A/B) = 1/4$ and $P(B/A) = 1/2$, then

(a) A and B are independent
(b) A and B are mutually exclusive
(c) $P(A'/B) = 3/4$
(d) $P(B'/A') = 1/2$

83. A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II, III are p, q and 1/2 respectively. If the probability that the student is successful is 1/2 then

(a) $p = 1, q = 0$ (b) $p = 2/3, q = 1/2$
(c) $p = 3/5, q = 2/3$
(d) there are infinitely many values of p and q.

84. A bag contains 14 balls of two colours, the number of balls of each colour being the same. 7 balls are drawn at random one by one. The ball in hand is returned to the bag before each new drawn. If the probability that at least 3 balls of each colour are drawn is p then

(a) $p > \frac{1}{2}$ (b) $p = \frac{1}{2}$

(c) $p < 1$ (d) $p < \frac{1}{2}$

85. All the spades are taken out from a pack of cards. From these cards; cards are drawn one by one without replacement till the ace of spades comes. The probability that the ace comes in the 4th draw is

(a) 1/13 (b) 12/13
(c) 4/13 (d) None of these

86. Let X be a set containing n elements. If two subsets A and B of X are picked at random, the probability that A and B have the same number of elements is

(a) $\dfrac{{}^{2n}C_n}{2^{2n}}$ (b) $\dfrac{1}{{}^{2n}C_n}$

(c) $\dfrac{1.3.5.\ldots(2n-1)}{2^n \cdot n!}$ (d) $\dfrac{3^n}{4^n}$

87. A die is thrown $2n + 1$ times, $n \in N$. The probability that faces with even numbers show odd number of times is

(a) $\dfrac{2n+1}{2n+3}$ (b) less than $\frac{1}{2}$

(c) greater than 1/2 (d) None of these

88. If $\overline{E}$ and $\overline{F}$ are the complementary events of the events E and F respectively then

(a) $P(E/F) + P(\overline{E}/F) = 1$
(b) $P(E/F) + P(E/\overline{F}) = 1$

(c) $P(\overline{E}/F) + P(E/\overline{F}) = 1$

(d) $P(E/F) + P(\overline{E}/F) = 1$

89. A natural number x is chosen at random from the first one hundred natural numbers. The probability that $\frac{(x-20)(x-40)}{(x-30)} < 0$ is

(a) 1/50 (b) 3/50

(c) 7/25 (d) 9/50

90. Three six faced fair dice are thrown together. The probability that the sum of the numbers appearing on the dice is k $(3 < k < 8)$ is

(a) $\frac{(k-1)(k-2)}{432}$ (b) $\frac{k(k-1)}{432}$

(c) ${}^{k-1}C_2 \times \frac{1}{216}$ (d) $\frac{k^2}{432}$

Practice Test

M.M. : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. A four digit number (numered from 0000 to 9999) is said to be lucky if sum of its first two digits is equal to sum of its last two digits. If a four digit number is picked up at random, the probability that it is lucky number is

(a) 1·67 (b) 2·37

(c) 0·067 (d) 0·37

2. A number is chosen at random from the numbers 10 to 99. By seing the number a man will laugh if product of the digits is 12. If he chosen three numbers with replacement then the probability that he will laugh at least once is

(a) $1 - \left(\frac{31}{45}\right)^3$ (b) $1 - \left(\frac{43}{45}\right)^3$

(c) $1 - \left(\frac{42}{43}\right)^3$ (d) $1 - \left(\frac{41}{45}\right)^3$

3. If X follows a binomial distribution with parameters $n = 8$ and $p = 1/2$ then $p(|x-4| < 2) =$

(a) 121/128 (b) 119/128

(c) 117/128 (d) 115/128

4. Two numbers b and c are chosen at random (with replacement from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9). The probability that $x^2 + bx + c > 0$ for all $x \in R$ is

(a) 17/123 (b) 32/81

(c) 82/ 125 (d) 45/143

5. Suppose n boys and m girls take their seats randomly round a circle. The probability of their sitting is $({}^{2n-1}C_n)^{-1}$ when

(a) no two boys sit together

(b) no two girls sit together

(c) boys and girls sit alternatively

(d) all the boys sit together

6. The probabilities of different faces of a biased dice to appear are as follows :

Face number	1	2	3	4	5	6
Probability	0 1	0·32	0·21	0 15	0 05	0·17

The dice is thrown and it is known that either the face number 1 or 2 will appear. Then the probability of the face number 1 to appear is

(a) 5/21 (b) 5/13

(c) 7/23 (d) 3/10

7. A card is selected at random from cards numbered as 00, 01, 02, ...,99. An event is said to have occured. If product of digits of the card number is 16. If card is selected 5 times with replacement each time, then the probability that the event occurs exactly three times is

(a) ${}^5C_3\left(\frac{3}{100}\right)^2\left(\frac{97}{100}\right)^3$

(b) ${}^5C_3\left(\frac{3}{100}\right)^3\left(\frac{97}{100}\right)^2$

(c) ${}^5C_3\left(\frac{\cdot 3}{100}\right)^3\left(\frac{9\cdot 7}{100}\right)^2$

(d) $20\,(0{\cdot}03)^3\,(0{\cdot}97)^2$

8. Let A, B, C be three mutually independent events. Consider the two statements S_1 and S_2.

$S_1 : A$ and $B \cup C$ are independent

$S_2 : A$ and $B \cap C$ are independent then

(a) Both S_1 and S_2 are true

(b) Only S_1 is true

(c) Only S_2 is true

(d) Neither S_1 nor S_2 is true

9. Sixteen players $P_1, P_2, ..., P_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assuming that all the players are of equal strength, the probability that exactly one of the two players P_1 and P_2 is among the eight winners is

(a) 4/15 (b) 7/15

(c) 8/15 (d) 17/30

10. The probabilities that a student in Mathematics, Physics and Chemistry are α, β and γ respectively. Of these subjects, a student has a 75% chance of passing in at least one, a 50% chance of passing in at least two, and a 40% chance of passing in exactly two subjects. Which of the following relations are true ?

(a) $\alpha + \beta + \gamma = 19/20$

(b) $\alpha + \beta + \gamma = 27/20$

(c) $\alpha\beta\gamma = 1/10$ (d) $\alpha\beta\gamma = 1/4$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (c)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (c)
7. (b)	**8.** (d)	**9.** (c)	**10.** (b)	**11.** (b)	**12.** (d)
13. (b)	**14.** (d)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (d)
19. (c)	**20.** (b)	**21.** (c)	**22.** (d)	**23.** (b)	**24.** (b)
25. (b)	**26.** (c)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (d)
31. (d)	**32.** (d)	**33.** (c)	**34.** (b)	**35.** (c)	**36.** (d)
37. (b)	**38.** (c)	**39.** (b)	**40.** (d)	**41.** (d)	**42.** (a)
43. (b)	**44.** (d)	**45.** (c)	**46.** (d)	**47.** (d)	**48.** (c)
49. (a)	**50.** (d)	**51.** (a)	**52.** (b)	**53.** (c)	**54.** (c)
55. (a)	**56.** (d)	**57.** (d)	**58.** (d)	**59.** (d)	**60.** (d)

Multiple Choice –II

61. (a), (d)	**62.** (a), (b), (c), (d)		**63.** (b)	**64.** (a)	**65.** (a), (b), (c)
66. (a)	**67.** (b)	**68.** (c)	**69.** (c)	**70.** (d)	**71.** (a)
72. (b), (c), (d)	**73.** (a)	**74.** (c)	**75.** (c)	**76.** (a)	**77.** (b)
78. (c)	**79.** (b)	**80.** (b)	**81.** (a), (b), (c), (d)		**82.** (a), (c), (d)
83. (a), (b), (c), (d)		**84.** (a)	**85.** (a)	**86.** (a, c)	**87.** (d)
88. (a, d)	**89.** (d)	**90.** (a, c)			

Practice Test

1. (c)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (a), (b), (c)	**6.** (a)
7. (b), (d)	**8.** (a)	**9.** (c)	**10.** (b), (c)		

8

LOGARITHMS AND THEIR PROPERTIES

1. Definition : e is the base of natural logarithm (Napier logarithm).

i.e., $ln\, x = \log_e x$

and $\log_{10} e$ is known as Napierian constant

i.e., $\log_{10} e = 0{\cdot}43429448 \ldots$

$\therefore$ $ln\, x = 2{\cdot}303 \log_{10} x$ $\left(\text{Since } ln\, x = \log_{10} x \,.\, \log_e 10 = \dfrac{1}{\log_{10} e} = 2{\cdot}30258509 \ldots\right)$

2. Properties :

(i) $a^k = x \Leftrightarrow \log_e x = k$; $a > 0,\ a \neq 1,\ x > 0$

(ii) $a^x = e^{x\, ln\, a},\ a > 0$

(iii) $n\,!\,(x) = 2n\pi\, i + ln\,(x)$; $x \neq 0,\ i = \sqrt{-1},\ x > 0$

(iv) $\log_a (mn) = \log_a m + \log_a n$; $a > 0,\ a \neq 1,\ m, n > 0$

(v) $\log_a (m/n) = \log_a m - \log_a n$; $a > 0,\ a \neq 1,\ m, n > 0$

(vi) $\log_a (1) = 0$; $a > 0,\ a \neq 1$ (vii) $\log_a a = 1$; $a > 0,\ a \neq 1$

(viii) $\log_a (m)^n = n \log_a (m)$; $a > 0,\ a \neq 1,\ m > 0$

(ix) $a^{\log_b (x)} = (x)^{\log_b (a)}$; $b \neq 1$; a, b, x an positive numbers.

(x) $a^{\log_a (x)} = x$; $a > 0,\ a \neq 1,\ x > 0$

(xi) $\log_b (x) = \left[\dfrac{\log_a (x)}{\log_a (b)}\right]$; $a \neq 1,\ b \neq 1$; a, b, x are positive numbers.

(xii) $\log_{a^k} (x) = \dfrac{1}{k} \log_a (x)$; $a > 0,\ a \neq 1,\ x > 0$

(xiii) $\log_a (x)^{2k} = 2k \log_a |x|$; $x > 0,\ k \neq 0$; $a > 0,\ a \neq 1$

(xiv) $\log_{a^{2k}} (x) = \dfrac{1}{2k} \log_{|a|} (x)$; $x > 0,\ k \neq 0$; $a > 0,\ a \neq 1$

(xv) $\log_a (x^2) \neq 2 \log_a (x)$ — Since domain of $\log_a (x^2)$ is $R \sim \{0\}$ and domain of $\log_a (x)$ is $(0, \infty)$ are not same and for $x < 0$, $\log_a x$ is imaginary.

(xvi) $\log_a x > 0$ if either $x > 1,\ a > 1$ or $0 < x < 1,\ 0 < a < 1$

(xvii) $\log_a x < 0$ if either $x > 1,\ 0 < a < 1$ or $0 < x < 1,\ a > 1$

(xviii) If $\log_a (x) > \log_a (y)$ then $x > y$ if $a > 1$ and $x < y$ if $0 < a < 1$

(xix) If $a > 1$, $\log_a (x) < k \Leftrightarrow 0 < x < a^k$ and $\log_a (x) > k \Leftrightarrow x > a^k$

(xx) If $0 < a < 1$, $\log_a (x) < k \Leftrightarrow x > a^k$ and $\log_a (x) > k \Leftrightarrow 0 < x < a^k$

(xxi) If $a^x = a^y$ then the following case hold :

(i) x and y can be any integer if $a = 1$

(ii) x and y can be any even integer if $a = -1$

(iii) x and y can be any real number if $a = 0$

(iv) $x = y$ if $a \neq 0, -1, +1$.

MULTIPLE CHOICE–I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d which ever is appropriate.

1. If $A = \log_2 \log_2 \log_4 256 + 2 \log_{\sqrt{2}} 2$ then $A =$

(a) 2 (b) 3

(c) 5 (d) 7

2. $7 \log\left(\dfrac{16}{15}\right) + 5 \log\left(\dfrac{25}{24}\right) + 3 \log\left(\dfrac{81}{80}\right)$ is equal to

(a) 0 (b) 1

(c) log 2 (d) log 3

3. For $y = \log_a x$ to be defined 'a' must be

(a) any positive real number

(b) any number (c) $> e$

(d) any positive real number $\neq 1$

4. If $\log_{10} 3 = 0{\cdot}477$, the number of digit in 3^{40} is
   (a) 18 (b) 19
   (c) 20 (d) 21

5. If $x = \log_3 5, y = \log_{17} 25$, which one of the following is correct ?
   (a) $x < y$ (b) $x = y$
   (c) $x > y$ (d) None of these

6. The domain of the function $\sqrt{(\log_{0\cdot5} x)}$ is
   (a) $(1, \infty)$ (b) $(0, \infty)$
   (c) $(0, 1]$ (d) $(0{\cdot}5, 1)$

7. The number $\log_2 7$ is
   (a) an integer (b) a rational number
   (c) an irrational number
   (d) a prime number

8. The value of
   $\dfrac{1}{\log_2 n} + \dfrac{1}{\log_3 n} + \ldots + \dfrac{1}{\log_{43} n}$ is
   (a) $\dfrac{1}{\log 43 ! \, n}$ (b) $\dfrac{1}{\log_{43} n}$
   (c) $\dfrac{1}{\log_{42} n}$ (d) $\dfrac{1}{\log_{43} n\,!}$

9. $\log_{10} \tan 1° + \log_{10} \tan 2° + \ldots + \log_{10} \tan 89° =$
   (a) 0 (b) 1
   (c) 27 (d) 81

10. If $\log_{12} 27 = a$ then $\log_6 16 =$
   (a) $2 \cdot \left(\dfrac{3-a}{3+a}\right)$ (b) $3 \cdot \left(\dfrac{3-a}{3+a}\right)$
   (c) $4\left(\dfrac{3-a}{3+a}\right)$ (d) $5\left(\dfrac{3-a}{3+a}\right)$

11. $\log_7 \log_7 \sqrt{7\sqrt{(7\sqrt{7})}} =$
   (a) $3 \log_2 7$ (b) $3 \log_7 2$
   (c) $1 - 3 \log_7 2$ (d) $1 - 3 \log_2 7$

12. If $\dfrac{\log x}{b-c} = \dfrac{\log y}{c-a} = \dfrac{\log z}{a-b}$ then $x^a . y^b . z^c =$
   (a) xyz (b) abc
   (c) 0 (d) 1

13. $\dfrac{1}{1 + \log_a bc} + \dfrac{1}{1 + \log_b ca} + \dfrac{1}{1 + \log_c ab} =$
   (a) 0 (b) 1
   (c) 2 (d) 3

14. If $(4)^{\log_9 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$ then $x =$
   (a) 2 (b) 3
   (c) 10 (d) 30

15. If x, y, z are in G.P. and $a^x = b^y = c^z$ then
   (a) $\log_b a = \log_c b$ (b) $\log_c b = \log_a c$
   (c) $\log_a c = \log_b a$ (d) $\log_a b = 2 \log_a c$

16. If $\dfrac{x\,(y+z-x)}{\log x} = \dfrac{y\,(z+x-y)}{\log y} = \dfrac{z\,(x+y-z)}{\log z}$
   then $x^y y^x = z^y y^z =$
   (a) $z^x x^z$ (b) $x^z y^x$
   (c) $x^y y^z$ (d) $x^x y^y$

17. If $\log_3 2, \log_3 (2^x - 5), \log_3 (2^x - 7/2)$ are in A.P. then $x =$
   (a) 1 (b) 2
   (c) 3 (d) 4

18. If $y = a^{\frac{1}{1 - \log_a x}}$ and $z = a^{\frac{1}{1 - \log_a y}}$ then $x =$
   (a) $a^{\frac{1}{1 + \log_a z}}$ (b) $a^{\frac{1}{2 + \log_a z}}$
   (c) $a^{\frac{1}{1 - \log_a z}}$ (d) $a^{\frac{1}{2 - \log_a z}}$

19. If $\log_{\sqrt{8}} b = 3\,\dfrac{10}{3}$ then $b =$
   (a) 2 (b) 8
   (c) 32 (d) 64

20. If $\log_{.3} (x - 1) < \log_{0.09} (x - 1)$ then x lies in the interval
   (a) $(-\infty, 1)$ (b) $(1, 2)$
   (c) $(2, \infty)$ (d) None of these

MULTIPLE CHOICE–II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

21. The value of $3^{\log_4 5} - 5^{\log_4 3}$ is
   (a) 0 (b) 1
   (c) 2 (d) None of these

22. The value of $\log_b a . \log_c b . \log_d c . \log_a d$ is
   (a) 0 (b) $\log abcd$
   (c) $\log 1$ (d) 1

23. If $x^{18} = y^{21} = z^{28}$ then $3, 3 \log_y x, 3 \log_z y, 7 \log_x z$ are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) A.G.P.

24. If $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > x$ then x be

(a) 2 (b) 3
(c) π (d) None of these

25. If $\dfrac{\log_2 x}{4} = \dfrac{\log_2 y}{6} = \dfrac{\log_2 z}{3k}$ and $x^3 y^2 z = 1$ then $k =$

(a) -8 (b) -4
(c) 0 (d) $\log_2\left(\dfrac{1}{256}\right)$

26. If $ln\left(\dfrac{a+b}{3}\right) = \left(\dfrac{ln\, a + ln\, b}{2}\right)$ then $\dfrac{a}{b} + \dfrac{b}{a} =$

(a) 1 (b) 3
(c) 5 (d) 7

27. If $n = 1983\,!$ then the value of
$$\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \ldots + \frac{1}{\log_{1983} n}$$
is equal to

(a) -2 (b) -1
(c) 0 (d) 1

28. If $\dfrac{\log a}{(b-c)} = \dfrac{\log b}{(c-a)} = \dfrac{\log c}{(a-b)}$ then
$$a^{b+c} . b^{c+a} . c^{a+b} =$$

(a) 0 (b) 1
(c) $a+b+c$ (d) $\log_b a . \log_c b \log_a c$

29. If $\log_3 (5 + 4 \log_3 (x-1)) = 2$, then $x =$

(a) 2 (b) 4
(c) 8 (d) $\log_2 16$

30. If $2x^{\log_4 3} + 3^{\log_4 x} = 27$ then $x =$

(a) 2 (b) 4
(c) 8 (d) 16

Practice Test

M.M. : 10 Time 15 Min.

(A) There are 5 parts in this question. Each part has one or more than one correct answer(s).

[5 × 2 = 10]

1. The interval of x in which the inequality
$$5^{1/4 \log^2 x} > 5x^{1/5 \log_5 x}$$

(a) $(0, 5^{-2\sqrt{5}}]$ (b) $[5^{2\sqrt{5}}, \infty)$
(c) both (a) & (b) (d) None of these

2. The solution set of the equation
$$\log_x 2 \log_{2x} 2 = \log_{4x} 2 \text{ is}$$

(a) $\{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$ (b) $\{1/2, 2\}$
(c) $\{1/4, 2^2\}$ (d) None of these

3. The least value of the expression
$$2 \log_{10} x - \log_x 0{\cdot}01 \text{ is}$$

(a) 2 (b) 4
(c) 6 (d) 8

4. The solution of the equation
$$\log_7 \log_5 (\sqrt{x+5} + \sqrt{x}) = 0 \text{ is}$$

(a) 1 (b) 3
(c) 4 (d) 5

5. The number of solutions of
$$\log_4 (x-1) = \log_2 (x-3) \text{ is}$$

(a) 3 (b) 1
(c) 2 (d) 0

Answers

Multiple Choice –I

1. (c)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (c)	**6.** (c)
7. (c)	**8.** (a)	**9.** (a)	**10.** (c)	**11.** (c)	**12.** (a)
13. (b)	**14.** (c)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (c)
19. (c)	**20.** (c)				

Multiple Choice –II

21. (a)	**22.** (a), (c)	**23.** (a)	**24.** (a)	**25.** (a), (d)	**26.** (d)
27. (d)	**28.** (b), (d)	**29.** (b)	**30.** (d)		

Practice Test

1. (c)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (b)

9

MATRICES

IMPORTANT DEFINITIONS, FORMULAE AND TECHNIQUES

§ 9.1 Definition

An $m \times n$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where the symbols a_{ij} represent any numbers (a_{ij} lies in the ith row (from top) and jth column (from left)).

Matrices represented by [ ], ( ), || ||

Note : If two matrices **A** and **B** are of the same order, then only their addition and subtraction is possible and these matrices are said to be **conformable** for addition or subtraction. On the other hand if the matrices **A** and **B** are of different orders then their addition and subtraction is not possible and these matrices are called **non-conformable** for addition and subtraction.

§ 9.2 Various Kinds of Matrices

(i) Idempotent Matrix : A square matrix **A** is called idempotent provided it satisfies the relation $\mathbf{A}^2 = \mathbf{A}$.

(ii) Periodic Matrix : A square matrix **A** is the least positive integer for which $\mathbf{A}^{k+1} = \mathbf{A}$, then k is said to be **period** of **A**. For $k = 1$, we get $\mathbf{A}^2 = \mathbf{A}$ and we called it to be **idempotent matrix.**

(iii) Nilpotent Matrix : A square matrix **A** is called Nilpotent matrix of order m provided it satisfies the relation $\mathbf{A}^k = \mathbf{0}$ and $\mathbf{A}^{-1} \neq 0$, where k is positive integer and **0** is null matrix and k is the order of the nilpotent matrix **A**.

(iv) Involutory Matrix : A square matrix **A** is called involutory provided it satisfies the relation $\mathbf{A}^2 = \mathbf{I}$, where **I** is identity matrix.

(v) Symmetric Matrix : A square matrix will be called symmetric if for all values of i and j, *i.e.*, $a_{ij} = a_{ji}$ or $\mathbf{A}' = \mathbf{A}$

(vi) Skew Symmetric Matrix : A square matrix is called skew symmetric matrix if (i) $a_{ij} = -a_{ji}$ for all values of i and j. (ii) All diagonal elements are zero, or $\mathbf{A}' = -\mathbf{A}$.

Note : Every square matrix can be uniquely expressed as the sum of symmetric and skew symmetric matrix.

i.e., $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}') + \frac{1}{2}(\mathbf{A} - \mathbf{A}')$, where $\frac{1}{2}(\mathbf{A} + \mathbf{A}')$ and $\frac{1}{2}(\mathbf{A} - \mathbf{A}')$

are symmetric and skew symmetric parts of **A**

(vii) Orthogonal Matrix : A square matrix **A** is called an orthogonal matrix if the product of the matrix **A** and its transpose **A′** is an identity matrix.

i.e., $\mathbf{AA}' = \mathbf{I}$

Note (i) If $\mathbf{AA}' = I$ then $\mathbf{A}^{-1} = \mathbf{A}'$

Note (ii) If **A** and **B** are orthogonal then **AB** is also orthogonal.

(viii) Complex Conjugate (or Conjugate) of a Matrix : if a matrix **A** is having complex numbers as its elements, the matrix obtained from **A** by replacing each element of **A** by its conjugate ($\overline{a \pm ib} = a \mp ib$) is called the conjugate of matrix **A** and is denoted by $\bar{\mathbf{A}}$.

(ix) Hermitian Matrix : A square matrix **A** such that $\overline{\mathbf{A}'} = \mathbf{A}$ is called Hermitian matrix, provided $a_{ij} = \bar{a}_{ji}$ for all values of i and j or $\mathbf{A}^{\Theta} = \mathbf{A}$.

(x) Skew-Hermitian Matrix : A square matrix **A** such that $\overline{\mathbf{A}'} = -\mathbf{A}$ is called skew-hermitian matrix, provided $a_{ij} = -\bar{a}_{ij}$ for all values of i and j or $\mathbf{A}^{\Theta} = -\mathbf{A}$.

(xi) Unitary Matrix : A square matrix **A** is called a unitary matrix if $\mathbf{A}\mathbf{A}^{\Theta} = \mathbf{I}$, where **I** is an identity matrix and $\mathbf{A}^{\Theta}$ is the transposed conjugate of **A**.

Properties of Unitary Matrix

(i) If **A** is unitary matrix, then **A′** is also unitary.

(ii) If **A** is unitary matrix, then $\mathbf{A}^{-1}$ is also unitary.

(iii) If **A** and **B** are unitary matrices then **AB** is also unitary.

§ 9.3 Properties of adjoint A

(a) If **A** be n rowed square matrix then

$$(\text{adj } \mathbf{A})\,\mathbf{A} = \mathbf{A}\,(\text{adj } \mathbf{A}) = |\,\mathbf{A}\,| \,.\, I_n$$

i.e., the product of a matrix and its adjoint is commutative.

Deductions of a:

Deducation 1. If **A** is n rowed square singular matrix then

$$(\text{adj } \mathbf{A})\,\mathbf{A} = \mathbf{A}\,(\text{adj } \mathbf{A}) = \mathbf{0} \text{ (null matrix)}$$

since for singular matrix, $|\,\mathbf{A}\,| = 0$.

Deduction 2. If **A** is n rowed square non-singular matrix, then

$$|\,\text{adj } \mathbf{A}\,| = |\,\mathbf{A}\,|^{n-1}$$

since for singular matrix, $|\,\mathbf{A}\,| \neq 0$.

(b) Adj (**AB**) = (Adj **B**) · (Adj **A**)

(c) (Adj **A**)′ = Adj **A′**

(d) adj (adj **A**) = $|\,\mathbf{A}\,|^{n-2}$ **A**, where **A** is a non-singular matrix.

(e) $|\{Adj\,(Adj)\,\mathbf{A}\}| = |\,\mathbf{A}\,|^{(n-1)^2}$, where **A** is a non-singular matrix.

(f) Adjoint of a diagonal matrix is a diagonal matrix.

(g) $\det(nA) = n^n \det(A)$

Note : Inverse of a non-singular diagonal matrix :

If $$\mathbf{A} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \text{ such that } |\,\mathbf{A}\,| \neq 0$$

$$\therefore \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

§ 9.4 Types of Equations

(1) When system of equations is non-homogeneous :

(i) If $|\,\mathbf{A}\,| \neq 0$, then the system of equations is consistent and has a unique solution give by $\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$.

(ii) If $|\,\mathbf{A}\,| = 0$ and (adj **A**) · **B** ≠ 0, then the system of equations is inconsistent and has no solutions.

(iii) If $|\,\mathbf{A}\,| = 0$ and (adj **A**) · **B** = **O** then the system of equations is consistent and has an infinite number of solutions.

(2) When system of equations is homogeneous :

(i) If $|A| \neq 0$, the system of equations have only trivial solution and it has one solution.
(ii) If $|A| = 0$, the system of equations has non-trivial solution and it has infinite solutions.
(iii) If No. of equations < No. of unknowns, then it has non trivial solution.

Note : Non-homogeneous linear equations also solved by cramer's rule this method has been discussed in the chapter on determinants.

§ 9.5 Rank of Matrix

The rank of a matrix is said to be r if
(i) If has at least minors of order r is different from zero.
(ii) All minors of **A** of order higher than r are zero.
The rank of **A** is denoted by ρ (**A**).

Note 1. The rank of a zero matrix is zero and the rank of an identity matrix of order n is n.
Note 2. The rank of a matrix in echelon form is equal to the number of non-zero rows **of** the matrix.
Note 3. The rank of a non-singular matrix ($|A| \neq 0$) of order n is n.

§ 9.6 Types of Linear Equations

(1) Consistent Equations : If Rank of **A** = Rand of **C**
(i) **Unique Solution :** Rank of **A** = Rank of **C** = n
where n = number of unknowns
(ii) **Infinite Solution :** Rank of **A** = Rank of **C** = r
where $r < n$
(2) Inconsistent Equations : *i.e.*, no solutions.
Rank of **A** ≠ Rank of **C**.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. Let A and B be two matrices, then
(a) $AB = BA$ (b) $AB \neq BA$
(c) $AB < BA$ (d) $AB > BA$

2. Let A and B be two matrices such that $A = 0, AB = 0$, then equation always implies that
(a) $B = 0$ (b) $B \neq 0$
(c) $B = -A$ (d) $B = A'$

3. In matrices :
(a) $(A+B)^2 = A^2 + 2AB + B^2$
(b) $(A+B)^2 = A^2 + B^2$
(c) $(A+B)^2 \neq A^2 + 2AB + B^2$
(d) $(A+B)^2 = A^2 + 2BA + B^2$

4. The characteristic of an orthogonal matrix A is
(a) $A^{-1} \cdot A = I$ (b) $A \cdot A^{-1} = I$
(c) $A' \cdot A^{-1} = I$ (d) $A \cdot A' = I$

5. The rank of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is equal to
(a) 4 (b) 3
(c) 5 (d) 1

6. $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$
(a) $x = 3, y = -1$ (b) $x = 2, y = 5$
(c) $x = 1, y = -1$ (d) $x = -1, y = 1$

7. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, which of the following result is true ?
(a) $A^2 = I$ (b) $A^2 = -I$
(c) $A^2 = 2I$ (d) None of these

8. With 1 ω, ω^2 as cube roots of unity, inverse of which of the following matrices exists ?
(a) $\begin{bmatrix} 1 & \omega \\ \omega & \omega^2 \end{bmatrix}$ (b) $\begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix}$
(c) $\begin{bmatrix} \omega & \omega^2 \\ \omega^2 & 1 \end{bmatrix}$ (d) None of these

9. If A is an orthogonal matrix, then A^{-1} equals

(a) A (b) A′
(c) A^2 (d) None of these

10. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & -3 & 8 \\ 9 & 2 & 16 \end{bmatrix}$, then trace of A is,
(a) 17 (b) 25
(c) 8 (d) 15

11. If A is a square matrix of order $n \times n$, then adj (adj A) is equal to
(a) $|A|^n A$ (b) $|A|^{n-1} A$
(c) $|A|^{n-2} A$ (d) $|A|^{n-3} A$

12. If A is a square matrix, then adj A^T – (adj A)^T is equal to
(a) 2 | A | (b) 2 | A | 1
(c) Null (d) Unit matrix

13. If $A = [a_{ij}]_{m \times n}$ is a matrix of rank r, then
(a) $r = \min(m, n)$ (b) $r > \min(m, n)$
(c) $r \le \min(m, n)$ (d) None of these

14. If A is an orthogonal matrix, then
(a) $|A| = 0$ (b) $|A| = \pm 1$
(c) $|A| = \pm 2$ (d) None of these

15. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is
(a) idempotent (b) nilpotent
(c) involutory (d) orthogonal

16. Matrix theory was introduced by
(a) Cauchy-Riemann
(b) Caley-Hamilton
(c) Newton
(d) Cacuchy-Schwar

17. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ then $A^{-1} =$
(a) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ (b) $\begin{bmatrix} a^2 & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & ac \end{bmatrix}$
(c) $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$ (d) $\begin{bmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{bmatrix}$

18. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then adj A =
(a) A (b) A^T
(c) 3 A (d) 3 A^T

19. The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix reflection in the line
(a) $x = 1$ (b) $x + y = 1$
(c) $y - 1$ (d) $x = y$

20. If I_n is the identity matrix of order n, then $(I_n)^{-1} =$
(a) does not exist (b) I_n
(c) 0 (d) $n\, I_n$

21. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & x \end{bmatrix}$ is an idempotent matrix then $x =$
(a) –5 (b) –1
(c) –3 (d) –4

22. If A is non-singular matrix, then Det $(A^{-1}) =$
(a) Det $\left(\frac{1}{A^2}\right)$ (b) $\frac{1}{\text{Det}(A^2)}$
(c) Det $\left(\frac{1}{A}\right)$ (d) $\frac{1}{\text{Det}(A)}$

23. The matrix $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 5 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is a nilpotent matrix of index
(a) 1 (b) 2
(c) 3 (d) 4

24. If A is a skew-symmetric matrix, then trace of A is
(a) –5 (b) 0
(c) 24 (d) 9

25. If the matrix $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & a \end{bmatrix}$ is unitary, then $a =$
(a) –2 (b) –1
(c) 0 (d) 1

26. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ and $AA' = I$, then

(a) –5 (b) –4
(c) –3 (d) –2

27. The sum of two idepotent matrices **A** and **B** is dempotent if **AB** = **BA** =
(a) 4 (b) 3
(c) 2 (d) 0

28. The rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ is equal to
(a) 1 (b) 2
(c) 3 (d) None of these

29. If **A, B, C** are non singular $n \times n$ matrices, then $(\mathbf{ABC})^{-1}$ =
(a) $A^{-1}B^{-1}C^{-1}$ (b) $C^{-1}B^{-1}A^{-1}$
(c) $A^{-1}C^{-1}B^{-1}$ (d) $B^{-1}C^{-1}A^{-1}$

30. If $\mathbf{A}^2 + \mathbf{A} - \mathbf{I} = \mathbf{0}$, then $\mathbf{A}^{-1}$ =
(a) $\mathbf{A} - \mathbf{I}$ (b) $\mathbf{I} - \mathbf{A}$
(c) $\mathbf{I} + \mathbf{A}$ (d) None of these

MULTIPLE CHOICE–II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

31. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then
(a) $\mathbf{A}^3 = 9\,\mathbf{A}$ (b) $\mathbf{A}^3 = 27\,\mathbf{A}$
(c) $\mathbf{A} + \mathbf{A} = \mathbf{A}^2$ (d) $\mathbf{A}^{-1}$ does not exist

32. $\mathbf{A} = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, then x
(a) 3 (b) 5
(c) 2 (d) 4

33. Let a, b, c be positive real numbers. The following system of equations in x, y and z
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has
(a) no solution
(b) unique solution
(c) infinitely many solutions
(d) finitely many solutions

34. If the matrix $\mathbf{A} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & \lambda \end{bmatrix}$ is singular, then λ =
(a) 3 (b) 4
(c) 2 (d) 5

35. If **A** is a 3×3 matrix and det (3 **A**) $= k$ {det (**A**)}, k =
(a) 9 (b) 6
(c) 1 (d) 27

36. If $\mathbf{A} = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x)$, then $\mathbf{A}^{-1}$ =
(a) $f(-x)$ (b) $f(x)$
(c) $-f(x)$ (d) $-f(-x)$

37. For all values of λ, the rank of the matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ \lambda & 8 & 8\lambda - 6 \\ 1 + \lambda^2 & 8\lambda + 4 & 2\lambda + 21 \end{bmatrix}$$
(a) for $\lambda = 2$, ρ (**A**) = 1
(b) for $\lambda = -1$, ρ (**A**) = 2
(c) for $\lambda \neq 2, -1$, ρ (**A**) = 3
(d) None of these

38. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then $|3AB|$ equals
(a) –9 (b) –91
(c) –27 (d) 81

39. The equations $2x + y = 5$, $x + 3y = 5$, $x - 2y = 0$ have
(a) no solution
(b) one solution
(c) two solutions
(d) infinity many solutions

40. If **A** is 3×4 matrix **B** is a matrix such **A′B** and **BA′** are both defined. Then **B** is of the type
(a) 3×4
(b) 3×3
(c) 4×4
(d) 4×3

Practice Test

M.M : 20 Time : 30 Min

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. If $\mathbf{A} = [a\ b]$, $\mathbf{B} = [-b\ -a]$ and $\mathbf{C} = \begin{bmatrix} a \\ -a \end{bmatrix}$, then the coorect statement is

(a) $\mathbf{A} = -\mathbf{B}$

(b) $\mathbf{A} + \mathbf{B} = \mathbf{A} - \mathbf{B}$

(c) $\mathbf{AC} = \mathbf{BC}$

(d) $\mathbf{CA} = \mathbf{CB}$

2. If $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, then $\mathbf{A}^{-1} =$

(a) $\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

3. If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 4 & 6 \\ 4 & 1 & 2 \\ -5 & -1 & 1 \end{bmatrix}$ then

(a) $\mathbf{A} + \mathbf{B}$ exists (b) $\mathbf{AB}$ exists

(c) $\mathbf{BA}$ exists (d) None of these

4. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$ is a

(a) rectangular matri

(b) singular matrix

(c) square matrix

(d) non singular matrix

5. $\mathbf{A}$ and $\mathbf{B}$ be 3×3 matrices. Then $|\mathbf{A} - \mathbf{B}| = 0$ implies

(a) $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$

(b) $|\mathbf{A}| = 0$ and $|\mathbf{B}| = 0$

(c) $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$

(d) $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$

6. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, then $\mathbf{A}^2$ is equal to

(a) $\mathbf{A}$ (b) $-\mathbf{A}$

(c) Null matrix (d) $\mathbf{I}$

7. If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, the value of X^n is

(a) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (b) $\begin{bmatrix} 2n+n & 5-n \\ n & -n \end{bmatrix}$

(c) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (d) None of these

8. Matrix $\mathbf{A}$ such that $\mathbf{A}^2 = 2\,\mathbf{A} - \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Then for $n > 2$, $\mathbf{A}^n =$

(a) $n\,\mathbf{A} - (n-1)\,\mathbf{I}$

(b) $n\,\mathbf{A} - \mathbf{I}$

(c) $2^{n-1}\,\mathbf{A} - (n-1)\,\mathbf{I}$U

(d) $2^{n-1}\,\mathbf{A} - \mathbf{I}$

9. For the equations : $x + 2y + 3z = 1$, $2x + y + 3z = 2$, $5x + 5y + 9z = 4$,

(a) there is only one solution

(b) there exists infinitely many solutions

(c) there is no solution

(d) None of these

10. Consider the system of equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$ if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

then the system has

(a) more than two solutions

(b) one trivial and one non-trivial solutions

(c) no solution

(d) only trivial solution (0, 0, 0)

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (b)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (b)	**6.** (a)
7. (b)	**8.** (d)	**9.** (b)	**10.** (d)	**11.** (c)	**12.** (c)
13. (c)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (d)
19. (d)	**20.** (b)	**21.** (c)	**22.** (d)	**23.** (b)	**24.** (b)
25. (b)	**26.** (c)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (c)

Multiple choice–II

31. (a), (c), (d)	**32.** (b)	**33.** (b)	**34.** (a)	**35.** (d)	**36.** (a)
37. (a), (b), (c)	**38.** (a)	**39.** (b)	**40.** (a)		

Practice Test

1. (c)	**2.** (b)	**3.** (c)	**4.** (c), (d)	**5.** (d)	**6.** (d)
7. (d)	**8.** (a)	**9.** (a)	**10.** (a)		

CALCULUS

10

FUNCTIONS

§ 10.1. Formulas for the Domain of a Function

1. Domain $(f(x) \pm g(x))$ = Domain $f(x) \cap$ Domain $g(x)$.
2. Domain $(f(x).g(x))$ = Domain $f(x) \cap$ Domain $g(x)$.
3. Domain $\left(\dfrac{f(x)}{g(x)}\right)$ = Domain $f(x) \cap$ Domain $g(x) \cap \{x : g(x) \neq 0\}$
4. Domain $\sqrt{f(x)}$ = Domain $f(x) \cap \{x : f(x) > 0\}$.
5. Domain (fog) = Domain $(g(x)$, where fog is defined by $fog(x) = f\{g(x)\}$.

Domain and Range of Inverse Trigonometric Functions

	Inverse Trigonometric functions	Domain (x)	Range (y) (Principal value)
(i)	$y = \sin^{-1}x$	$-1 < x < 1$	$-\pi/2 < y < \pi/2$
(ii)	$y = \cos^{-1}x$	$-1 < x < 1$	$0 < y < \pi$
(iii)	$y = \tan^{-1}x$	R	$-\pi/2 < y < \pi/2$
(iv)	$y = \cot^{-1}x$	R	$0 < y < \pi$
(v)	$y = \sec^{-1}x$	$x < -1$ $x > 1$	$\pi/2 < y < \pi$ $0 < y < \pi/2$
(vi)	$y = \text{cosec}^{-1}x$	$x \leq -1$ $x > 1$	$-\pi/2 \leq y < 0$ $0 < y < \pi/2$

§10.2. Odd and Even Function

(i) A function is an odd function if $f(-x) = -f(x)$ for all x.
(ii) A function is an even function if $f(-x) = f(x)$ for all x.

Extension of a function

If a function $f(x)$ is defined on the interval $[0, a]$, it can be extend on $[-a, a]$ so that $f(x)$ is either even or odd function on the interval $[-a, a]$.

(i) Even Extension : If a function $f(x)$ is defined on the interval $[0, a]$

$$0 < x < a \Rightarrow -a < -x < 0 \therefore -x \in [-a, 0]$$

We define $f(x)$ in the interval $[-a, 0]$ such that $f(x) = f(-x)$. Let g be the even extension, then

$$g(x) = \begin{cases} f(x,); & x \in [0, a] \\ f(-x); & x \in [-a, 0] \end{cases}$$

(ii) Odd Extension : If a function $f(x)$ is defined on the interval $[0, a]$

$$0 < x < a \Rightarrow -a < -x < 0 \therefore -x \in [-a, 0]$$

We define $f(x)$ in the interval $[-a, 0]$ such that $f(x) = -f(-x)$. Let g be the odd extension, then

$$g(x) = \begin{cases} f(x,); & x \in [0, a] \\ -f(-x); & x \in [-a, 0] \end{cases}$$

§ 10.3. Periodic Function

A function $f(x)$ is said to be a periodic function of x, if there exist a positive real number T such that $f(x+T) = f(x)$ for all x.

The smallest value of T is called the period of the function.

If positive value of T independent of x then $f(x)$ is periodic function and if the value of T depends upon x, then $f(x)$ is non-periodic.

§ 10.4. Methods to Find Period of a Periodic Function

(i) $\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$ are periodic functions with period 2π and π according as n be odd or even.

(ii) $\tan^n x$, $\cot^n x$ are periodic functions with period π, n even or odd.

(iii) $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, $|\text{cosec } x|$ are periodic functions with period π.

(iv) $|\sin x| + |\cos x|$, $|\tan x| + |\cot x|$, $|\sec x| + |\text{cosec } x|$ are periodic functions with period $\pi/2$.

(v) If $f(x)$ is a periodic with period T, then the function $f(ax+b)$ is periodic with period $\frac{T}{|a|}$.

(vi) If $f(x)$, $g(x)$ and $h(x)$ an periodic with periods T_1, T_2, T_3 respectively then period of

$$F(x) = af(x) \pm bg(x) \pm ch(x) \qquad \text{where } a, b, c \text{ are constant}$$
$$= \{\text{L.C.M. of } \{T_1, T_2, T_3\},$$

Note : $\text{L.C.M.}\left\{\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}\right\} = \frac{\text{L.C.M. of } \{a, b, c\}}{\text{H.C.F. of } \{a_1, b_1, c_1\}}$

§ 10.5. Invertible Function

Let $f: A \to B$ be a One-One and Onto function then their exists a unique function

$$g: B \to A$$

such that
$$f(x) = y \Leftrightarrow g(y) = x, \qquad \{g(y) = f^{-1}(y)\}$$
$$\forall \; x \in A \text{ and } y \in B$$

Then g is said to be inverse of f.

and $fof^{-1} = f^{-1}of = I$, I is an identity function

$$\Rightarrow \qquad (fof^{-1})x = I(x) = x$$

$$\Rightarrow \qquad f\{f^{-1}(x)\} = x$$

Note : If A and B are two sets having m and n elements such that $1 \le n \le m$ then

(i) Number of functions from A to $B = n^m$

(ii) Number of onto (or surjection) functions from A to B

$$= \sum_{r=1}^{n} (-1)^{n-r} \cdot {}^nC_r \, r^m$$

(iii) Number of one-one onto mapping or bijection $= n!$

(If A and B have same number of elements say n)

§ 10.6. Signum Function

The signum function f is defined as

$$Sgn\, f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

as x shown in the Fig 10.1

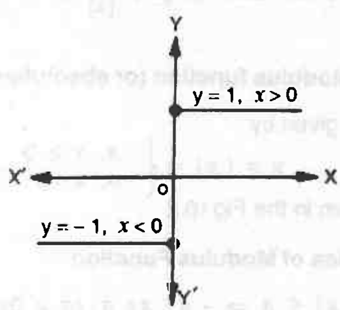


Fig. 10.1

§ 10.7. Greatest Integer Function

$[x]$ denotes the greatest integer less than or equal to x. *i.e.* $[x] < x$..

Thus $[3{\cdot}5778] = 3, \quad [0{\cdot}87] = 0, \quad [5] = 5$

$[-8{\cdot}9728] = -9, \quad [-0{\cdot}6] = -1.$

In general if n is an integer and x is any real number between n and $(n+1)$

i.e., $n < x < n+1$

then $[x] = n.$

Properties of Greatest Integer function

(i) If $f(x) = [x+n]$, where $n \in I$ and $[\,.\,]$ denotes the greatest integer function, then

$$f(x) = n + [x]$$

(ii) $x = [x] + \{x\}$, $[\,.\,]$ and $\{\,\}$ denote the integral and fractional part of x respectively

Least Integer Function

(x) or $[x]$ denotes the least integer function which is greater than or equal to x. It is also known as **ceiling of x.**

Thus, $(3{\cdot}578) = 4, (0{\cdot}87) = 1, (4) = 4,$

$[-8{\cdot}239] = -8, [-0{\cdot}7] = 0$

Properties of Least Integer function

(i) $(x+n) = (x) + n, \; n \in I$

(ii) $x = x + \{x\} - 1$, $\{x\}$ denotes the fractional part of x

(iii) $(-x) = -(x), \; x \in I$

(iv) $(-x) = -(x) + 1, \; x \notin I$

(v) $(x) > n \Rightarrow x > n - 1, \; n \in I$

(vi) $(x) > n \Rightarrow x > n, \; n \in I$

(vii) $(x) < n \Rightarrow x < n, \; n \in I$

(viii) $(x) < n \Rightarrow x < n - 1, \; n \in I$

(ix) $n_2 < (x) < n_1 \Rightarrow n_2 - 1 < x < n_1; \; n_1, n_2 \in I$

(x) $(x+y) > (x) + (y) - 1$

(xi) $\left(\dfrac{(x)}{n}\right) = \left(\dfrac{x}{n}\right), \; n \in N$

(xii) $\left(\dfrac{n+1}{2}\right)\left(\dfrac{n+2}{4}\right) + \left(\dfrac{n+4}{8}\right) + \left(\dfrac{n+8}{16}\right) + \ldots = 2n, \; n \in N$

(xiii) $(x) + \left(x + \dfrac{1}{n}\right) + \left(x + \dfrac{2}{n}\right) + \ldots + \left(x + \dfrac{n-1}{n}\right) = (nx) + n - 1, \; n \in N$

Fractional Part Function

It is denoted as $f(x) = \{x\}$ and defined as

(i) $\{x\} = f$ if $x = n + f$ where $n \in I$ and $0 < f < 1$

(ii) $\{x\} = x - [x]$

Notes : 1. For proper fraction $0 < f < 1$

2. Domain and Range of $\dfrac{1}{\{x\}}$ are R–I and (0, 1) respectively

§ 10.8. Modulus function (or absolute-value-function)

It is given by

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

it is shown in the Fig10.2

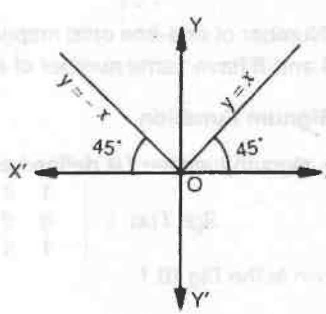


Fig. 10.2

Properties of Modulus Function

(i) $|x| < a \Rightarrow -a < x < a \; ; (a > 0)$

(ii) $|x| > a \Rightarrow x < -a$ or $x > a \; ; \; (a > 0)$

(iii) $|x+y| = |x|+|y| \Leftrightarrow x \geq 0$ & $y \geq 0$ or $x<0$ & $y<0$
(iv) $|x-y| = |x|-|y| \Rightarrow x>0$ & $|x|>|y|$
or $x<0$ & $y<0$ & $|x|>|y|$
(v) $|x \pm y| \leq |x|+|y|$
(vi) $|x \pm y| \geq ||x|-|y||$

§ 10.9 Form of the Function

By considering a general nth degree polynomial and writting the expression.

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

it can be proved by comparing the coefficients of $x^n, x^{n-1}, \ldots$, constant that the polynomial satisfying the above equation is either of the form

$f(x) = x^n + 1$ or $-x^n + 1$. Now

$f(3) = 28 \Rightarrow 3^n + 1 = 28 \Rightarrow n = 3$

$f(3) = -3^n + 1$ is not possible as $-3^n = 27$ is Not true for any value of n.

Hence $f(4) = 4^3 + 1 = 65$.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If $f(x) = -\dfrac{x|x|}{1+x^2}$ then $f^{-1}(x)$ equals

(a) $\sqrt{\dfrac{|x|}{1-|x|}}$ (b) $(Sgn\, x)\sqrt{\dfrac{|x|}{1-|x|}}$

(c) $-\sqrt{\dfrac{x}{1-x}}$ (d) None of these

2. Let $f: R \to R$ defined by $f(x) = \dfrac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$ then

(a) $f(x)$ is one-one but not onto
(b) $f(x)$ is neither one-one nor onto
(c) $f(x)$ is many one but onto
(d) $f(x)$ is one-one and onto

3. The function $f(x) = \lambda|\sin x| + \lambda^2|\cos x| + g(\lambda)$ has period equal to $\pi/2$ if λ is

(a) 2 (b) 1
(c) 3 (d) None of these

4. If f is decreasing odd function then f^{-1} is

(a) Odd and decreasing
(b) Odd and increasing
(c) Even and decreasing
(d) Even and increasing

5. The range of the function $f(x) = 3|\sin x| - 2|\cos x|$ is

(a) $[-2, \sqrt{13}]$ (b) $[-2, 3]$
(c) $[3, \sqrt{13}]$ (d) None of these

6. The domain of the function

$f(x) = \sqrt{\left(\dfrac{1}{\sin x} - 1\right)}$ is

(a) $\left(2n\pi, 2n\pi + \dfrac{\pi}{2}\right)$
(b) $(2n\pi, (2n+1)\pi)$
(c) $((2n-1)\pi, 2n\pi)$
(d) None of these

7. If $g(x) = [x^2] - [x]^2$, where [.] denotes the greatest integer function, and $x \in [0, 2]$, then the set of values of $g(x)$ is

(a) $\{-1, 0\}$ (b) $\{-1, 0, 1\}$
(c) $\{0\}$ (d) $\{0, 1, 2\}$

8. Which of the following functions is periodic with period π ?

(a) $f(x) = \sin 3x$ (b) $f(x) = |\cos x|$
(c) $f(x) = [x + \pi]$ (d) $f(x) = x\cos x$

where $[x]$ means the greatest integer not greater than x.

9. The domain of definition of

$f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$ is

(a) $(-\infty, \infty) - [-2, 2]$

(b) $(-\infty, \infty) - [-1, 1]$
(c) $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$
(d) None of these

10. Let $f: R \to R$ be a given function and $A \subset R$ and $B \subset R$ then
(a) $f(A \cup B) = f(A) \cup f(B)$
(b) $f(A \cap B) = f(A) \cap f(B)$
(c) $f(A^c) = [f(A)]^c$
(d) $f(A \setminus B) = f(A) \setminus f(B)$

11. The domain of the function $y = \underbrace{\log_{10} \log_{10} \log_{10} \ldots \log_{10}}_{n \text{ times}} x$ is
(a) $[10^n, \infty)$ (b) $(10^{n-1}, \infty)$
(c) $(10^{n-2}, \infty)$ (d) None of these

12. If $[x]$ and $\{x\}$ represent integral and fractional parts of x, then the value of $\sum_{r=1}^{2000} \frac{\{x+r\}}{2000}$ is
(a) x (b) $[x]$
(c) $\{x\}$ (d) $x + 2001$

13. If [.] denotes the greatest integer function then the value of $\sum_{r=1}^{100} \left[\frac{1}{2} + \frac{r}{100}\right]$ is
(a) 49 (d) 50
(c) 51 (d) 52

14. If $f(x)$ is a polynomial satisfying $f(x).f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then $f(4) =$
(a) 63 (b) 65
(c) 17 (d) None of these

15. If $f(x+y) = f(x) + f(y) - xy - 1$ for all x, y and $f(1) = 1$ then the number of solutions of $f(n) = n, n \in N$ is
(a) one (b) two
(c) three (d) None of these

16. The function
$$f(x) = \sin\left(\frac{\pi x}{n!}\right) - \cos\left(\frac{\pi x}{(n+1)!}\right) \text{ is}$$
(a) not periodic
(b) periodic, with period $2(n!)$
(c) periodic, with period $(n+1)$
(d) None of these

17. The value of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$ are
(a) $b = 2, c = 1$ (b) $b = 4, c = -1$
(c) $b = -1, c = 4$ (d) $b = -1, c = 1$

18. The value of the parameter α, for which the function $f(x) = 1 + \alpha x, \alpha \neq 0$ is the inverse of itself, is
(a) –2 (b) –1
(c) 1 (d) 2

19. Which of the following functions is even function
(a) $f(x) = \frac{a^x + 1}{a^x - 1}$ (b) $f(x) = x\frac{a^x - 1}{a^x + 1}$
(c) $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$ (d) $f(x) = \sin x$

20. If S is the set of all real x for which $1 - e^{(1/x)-1} > 0$, then $S =$
(a) $(-\infty, 0) \cup (1, \infty)$ (b) $(-\infty, \infty)$
(c) $(-\infty, 0] \cup [1, \infty)$ (d) None of these

21. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x . \cos\left(x + \frac{\pi}{3}\right)$ and $g(5/4) = 1$ then $(gof)\, x$ is
(a) a polynomial of the first degree in $\sin x, \cos x$
(b) a constant function
(a) a polynomial of the second degree in $\sin x, \cos x$
(d) None of these

22. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$ then $f^{-1}(x)$ is
(a) $\left(\frac{1}{2}\right)^{x(x-1)}$
(b) $\frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$
(c) $\frac{1}{2}(1 - \sqrt{1 + 4\log_2 x})$
(d) not defined

23. Let f be a function satisfying $2f(xy) = \{f(x)\}^y + \{f(y)\}^x$ and $f(1) = k \neq 1$, then $\sum_{r=1}^{n} f(r) =$

(a) $k^n - 1$ (b) k^n
(c) $k^n + 1$ (d) None of these

24. Which one of the following functions are periodic
(a) $f(x) = x - [x]$, where $[x] \le x$
(b) $f(x) = x \sin(1/x)$ for $x \ne 0, f(0) = 0$
(c) $f(x) = x \cos x$
(d) None of these

25. The domain of the function $f(x) = 1/\log_{10}(1 - x) + \sqrt{(x+2)}$ is
(a) $[-3, -2]$ excluding $(-2{\cdot}5)$
(b) $[0, 1]$, excluding $0{\cdot}5$
(c) $[-2, 1]$, excluding 0
(d) None of these

26. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$. Then
(a) $f(x+2) = f(x-2)$
(b) $f(2+x) = f(2-x)$
(c) $f(x) = f(-x)$
(d) None of these

27. The range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$ is
(a) $[-2, \infty)$ (b) $(-2, \infty)$
(c) $(6, \infty)$ (d) $[6, \infty)$

28. If $f: X \to Y$ defined by $f(x) = \sqrt{3}\sin x + \cos x + 4$ is one-one and onto, then Y is
(a) $[1, 4]$ (b) $[2, 5]$
(c) $[1, 5]$ (d) $[2, 6]$

29. If $f(x) = \cos^{-1}(x - x^2) + \sqrt{\left(1 - \frac{1}{|x|}\right)} + \frac{1}{[x^2 - 1]}$ then domain of $f(x)$ is (where [.] is the greatest integer)
(a) $\left[\sqrt{2}, \frac{1+\sqrt{5}}{2}\right]$ (b) $\left(\sqrt{2}, \frac{1-\sqrt{5}}{2}\right)$
(c) $\left[\sqrt{2}, \frac{1+\sqrt{5}}{2}\right]$ (d) None of these

30. The range of function $f: [0, 1] \to R, f(x) = x^3 - x^2 + 4x + 2\sin^{-1} x$ is
(a) $[-\pi - 2, 0]$ (b) $[2, 3]$
(c) $[0, 4 + \pi]$ (d) $(0, 2 + \pi]$

31. Let $f(x)$ be a function defined on $[0, 1]$ such that

$$f(x) = \begin{cases} x, & \text{if } x \in Q \\ 1 - x, & \text{if } x \notin Q \end{cases}$$

Then for all $x \in [0, 1]$ $f_o f(x)$ is
(a) Constant (b) $1 + x$
(c) x (d) None of these

32. If $f: R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$, then $f(2) - f(1)$
(a) $f(0)$ (b) $-f(0)$
(c) $f'(0)$ (d) $-f'(0)$

33. Let $f: R \to Q$ be a continuous function such that $f(2) = 3$ then
(a) $f(x)$ is always an even function
(b) $f(x)$ is always an odd function
(c) Nothing can be said about $f(x)$ being even or odd
(d) $f(x)$ is an increasing function

34. The greatest value of the function

$$f(x) = \cos\{x e^{[x]} + 2x^2 - x\}, x \in (-1, \infty)$$

where $[x]$ denotes the greatest integer less than or equal to x is
(a) 0 (b) 1
(c) 2 (d) 3

35. The period of $e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$ is
([.] denotes the greatest integer function)
(a) 2 (b) 1
(c) 0 (d) -1

36. If $f(x) = \sin^{-1}.\{4 - (x-7)^3\}^{1/5}$, then its inverse is
(a) $(4 - \sin^5 x)^{1/3}$ (b) $7 - (4 - \sin^5 x)^{1/3}$
(c) $(4 - \sin^5 x)^{2/3}$ (d) $7 + (4 - \sin^5 x)^{1/3}$

37. The period of $f(x) = \frac{1}{2}\left\{\frac{|\sin x|}{\cos s} + \frac{|\cos x|}{\sin x}\right\}$ is
(a) 2π (b) π
(c) $\pi/2$ (d) $\pi/4$

38. Given $f(x) = \frac{1}{(1-x)}, g(x) = f\{f(x)\}$ and $h(x) = f\{f\{f(x)\}\}$. Then the value of $f(x).\, g(x).\, h(x)$ is
(a) 0 (b) -1
(c) 1 (d) 2

39. The inverse of the function
$y = \log_a(x + \sqrt{x^2 + 1})\ (a > 0, a \ne 1)$ is

(a) $\frac{1}{2}(a^x - a^{-x})$
(b) not defined for all x
(d) defined for only positive x
(d) None of these

40. The domain of the function $f(x) = \sqrt{\sin^{-1}(\log_2 x)} + \sqrt{\cos(\sin x)} + \sin^{-1}\left(\frac{1+x^2}{2x}\right)$
(a) $\{x : 1 \le x \le 2\}$
(b) $\{1\}$
(c) Not defined for any value x
(d) $\{-1, 1\}$

41. Let $f : R \to R$ be a function defined by $f(x) = \frac{x^2 + 2x + 5}{x^2 + x + 1}$ is
(a) one-one and into
(b) one-one and onto
(c) many one and onto
(d) many one and into

42. Let $f(x) = \begin{cases} 1 + x, & 0 \le x \le 2 \\ 3 - x, & 2 < x \le 3 \end{cases}$ then $f_o f(x)$

(a) $= \begin{cases} 2 + x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \\ 4 - x, & 2 < x \le 3 \end{cases}$

(b) $= \begin{cases} 2 + x, & 0 \le x \le 2 \\ 4 - x, & 2 < x \le 3 \end{cases}$

(c) $= \begin{cases} 2 + x, & 0 \le x \le 2 \\ 2 - x, & 2 < x \le 3 \end{cases}$

(d) None of these

43. Let $f : R \to R$, $g : R \to R$ be two given functions susch that f is injective and g is surjective then which of the following is injective
(a) $g_o f$ (b) $f_o g$
(c) $g_o g$ (d) $f_o f$

44. The domain of $f(x) = \frac{1}{\sqrt{|\cos x| + \cos x}}$ is
(a) $[-2n\pi, 2n\pi]$
(b) $(2n\pi, \overline{2n+1}\,\pi)$
(c) $\left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2}\right)$
(d) $\left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right)$

45. The domain of the function $f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}P_{4x-5}$, where the symbols have their usual meanings, is the set
(a) $\{2, 3\}$ (b) $\{2, 3, 4\}$
(c) $\{1, 2, 3, 4, 5\}$ (d) None of these

46. If $f(x) = 3\sin\sqrt{\left(\frac{\pi^2}{16} - x^2\right)}$ then its range is
(a) $\left[-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]$ (b) $\left[0, \frac{3}{\sqrt{2}}\right]$
(c) $\left[-\frac{3}{\sqrt{2}}, 0\right]$ (d) None of these

47. The domain of $f(x) = \sqrt{x - 4 - 2\sqrt{(x-5)}} - \sqrt{x - 4 + 2\sqrt{x-5}}$ is
(a) $[-5, \infty)$
(b) $(-\infty, +2]$
(c) $[5, \infty), (-\infty, -2]$
(d) None of these

48. The period of $\frac{|\sin x| + |\cos x|}{|\sin x - \cos x|}$ is
(a) 2π (b) π
(c) $\pi/2$ (d) $\pi/4$

49. If [.] denotes the greatest integer function then the domain of the real valued function $\log_{[x+1/2]} |x^2 - x - 2|$ is
(a) $\left[\frac{3}{2}, \infty\right)$
(b) $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$
(c) $\left(\frac{1}{2}, 2\right) \cup (2, \infty)$
(d) None of these

50. Let $f(x) = \sin^2 x/2 + \cos^2 x/2$ and $g(x) = \sec^2 x - \tan^2 x$. The two functions are equal over the set
(a) ϕ
(b) R
(c) $R - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in I\right\}$
(d) None of these

MULTIPLE CHOICE –II

Each queston, in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

51. The domain of $f(x)$ is $(0, 1)$ therefore domain of $f(e^x) + f(ln\,|x|)$ is
(a) $(-1, e)$ (b) $(1, e)$
(c) $(-e, -1)$ (d) $(-e, 1)$

52. If $f : [-4, 0] \to R$ is defined by $e^x + \sin x$, its even extension to $[-4, 4]$ is given by
(a) $-e^{-|x|} - \sin|x|$ (b) $e^{-|x|} - \sin|x|$
(c) $e^{-|x|} + \sin|x|$ (d) $-e^{-|x|} + \sin|x|$

53. If $f(x) = \frac{x}{x^2+1}$ and $f(A) = \left\{ y : -\frac{1}{2} < y < 0 \right\}$ then set A is
(a) $[-1, 0)$ (b) $(-\infty, -1]$
(c) $(-\infty, 0)$ (d) $(-\infty, \infty)$

54. If $g(x)$ be a function defined on $[-1, 1]$ if the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $(x, g(x))$ is $\sqrt{3}/4$, then the function is
(a) $g(x) = \pm\sqrt{(1-x^2)}$
(b) $g(x) = -\sqrt{(1-x^2)}$
(c) $g(x) = \sqrt{(1-x^2)}$
(d) $g(x) = \sqrt{1+x^2}$

55. The period of the function
$f(x) = a^{\sin^2 x + \sin^2(x+\pi/3) + \cos x \cos(x+\pi/3)}$ is (where a is constant)
(a) 1 (b) $\pi/2$
(c) π (d) can not be determined

56. The domain of the function
$$f(x) = \sin^{-1}\left(\frac{2-|x|}{4}\right) + \cos^{-1}\left(\frac{2-|x|}{4}\right) + \tan^{-1}\left(\frac{2-|x|}{4}\right)$$ is
(a) $[0, 3]$ (b) $[-6, 6]$
(c) $[-1, 1]$ (d) $[-3, 3]$

57. Let f be a real valued function defined by $f(x) = \frac{e^x - e^{-|x|}}{e^x + e^{|x|}}$ then range of f is
(a) R (b) $[0, 1]$
(c) $[0, 1)$ (d) $[0, 1/2)$

58. Let $f(x) = 2x - \sin x$ and $g(x) = \sqrt[3]{x}$, then
(a) range of gof is R
(b) gof is one-one
(c) both f and g are one-one
(d) both f and g are onto

59. Let
$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ x^2 \sin\left(\frac{\pi}{x}\right) & \text{for } -1 < x < 1\ (x \neq 0) \\ x\,|x| & \text{for } x \geq 1 \text{ or } x \leq -1 \end{cases}$$
then
(a) $f(x)$ is an odd function
(b) $f(x)$ is an even function
(c) $f(x)$ is neither odd nor even
(d) $f'(x)$ is an even function

60. Which of the following function is periodic
(a) $Sgn(e^{-x})$
(b) $\sin x + |\sin x|$
(c) min $(\sin x, |x|)$
(d) $\left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] + 2[-x]$
($[x]$ denotes the greatest integer function)

61. Of the following functions defined from $[-1, 1]$ to $[-1, 1]$ select those which are not bijective
(a) $\sin(\sin^{-1} x)$ (b) $\frac{2}{\pi}\sin^{-1}(\sin x)$
(c) $(Sgn\,x)\,ln(e^x)$ (d) $x^3(Sgn\,x)$

62. If $[x]$ denotes the greatest integer less than or equal to x, the extreme values of the function $f(x) = [1 + \sin x] + [1 + \sin 2x] + [1 + \sin 3x] + \ldots + [1 + \sin nx]$, $n \in I^+$, $x \in (0, \pi)$ are
(a) $n - 1$ (b) n
(c) $n + 1$ (d) $n + 2$

63. If $f(x)$ is a polynomial function of the second degree such that $f(-3) = 6$, $f(0) = 6$ and $f(2) = 11$ then the graph of the function $f(x)$ cuts the ordinate $x = 1$ at the point
(a) $(1, 8)$ (b) $(1, -2)$
(c) 1, 4 (d) none of these

64. If $f(x+y, x-y) = xy$ then the arithmetic mean of $f(x, y)$ and $f(y, x)$ is

(a) x (b) y
(c) 0 (d) None of these

65. Under the condition ...., the domain of $f_1 + f_2$ is equal to $\text{dom} f_1 \cup \text{dom} f_2$.

(a) $\text{dom} f_1 \neq \text{dom} f_2$ (b) $\text{dom} f_1 = \text{dom} f_2$
(c) $\text{dom} f_1 > \text{dom} f_2$ (d) $\text{dom} f_1 < \text{dom} f_2$

66. Domain of $f'(x) = \sin^{-1}[2 - 4x^2]$ is ([.] denotes the greatest integer function)

(a) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right] - \{0\}$
(b) $\left[-\frac{\sqrt{3}}{2}, 0\right)$
(c) $\left[-\frac{\sqrt{3}}{2}, 0\right) \cup \left(0, \frac{\sqrt{3}}{2}\right]$
(d) $\left[-\frac{\sqrt{3}}{2}, 8\right]$

67. If $e^x + e^{f(x)} = e$ then for $f(x)$

(a) domain $= (-\infty, 1)$
(b) range $= (-\infty, 1)$
(c) domain $= (-\infty, 0]$
(d) range $= (-\infty, 1]$

68. Let $f(x) = \sec^{-1}[1 + \cos^2 x]$, where [.] denotes the greatest integer function, then

(a) the domain of f is R
(b) the domain of f is $[1, 2]$
(c) the range of f is $[1, 2]$
(d) the range of f is $\{\sec^{-1} 1, \sec^{-1} 2\}$

69. If the function $f: R \to R$ be such that $f(x) = x - [x]$, where [.] denotes the greatest integer function, then $f^{-1}(x)$ is

(a) $\frac{1}{x - [x]}$ (b) $[x] - x$
(c) not defined (d) None of these

70. The domain of the function $f(x) = \sqrt{(2 - |x|)} + \sqrt{(1 + |x|)}$

(a) [2, 6] (b) (–2, 6]
(c) [8, 12] (d) None of these

71. Let $f: R \to [0, \pi/2)$ be a function defined by $f(x) = \tan^{-1}(x^2 + x + a)$. If f is onto then a equals

(a) 0 (b) 1
(c) 1/2 (d) 1/4

72. Let $f(x) = \cos\sqrt{k}\,x$, where $k = [m] =$ the greatest integer $\leq m$, if the period of $f(x)$ is π then

(a) $m \in [4, 5)$ (b) $m = 4, 5$
(c) $m \in [4, 5]$ (d) None of these

73. Let $f(x) = [x]^2 + [x+1] - 3$, where $[x] \leq x$. Then

(a) $f(x)$ is a many-one and into function
(b) $f(x) = 0$ for infinite number of values of x
(c) $f(x) = 0$ for only two real values
(d) None of these

74. Domain of $\sin^{-1}[\sec x]$ ([.] is greatest integer less than or equal to x) is

(a) $\{(2n+1)\pi, (2n+9)\pi\} \cup \{[(2m-1)\pi, 2m\pi + \pi/3), m \in I\}$
(b) $\{2n\pi, n \in I\} \cup \{[2m\pi, (2m+1)\pi), m \in I\}$
(c) $\{(2n+1)\pi, n \in I\} \cup \{[2m\pi, 2m\pi, +\pi/3), m \in I\}$
(d) None of these

75. Let $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$. The domain of the function is

(a) $(-\infty, -1)$
(b) $(-1, 1)$
(c) $(1, \infty)$
(d) $(-\infty, \infty)$

Practice Test

M.M. : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s). [10 × 2 = 20]

1. The function $f(x) = \int_0^x \log_e\left(\frac{1-x}{1+x}\right) dx$ is

(a) an even function
(b) an odd function
(c) a periodic function
(d) None of these

2. If $f: R \to R, g: R \to R$ be two given functions then $f(x) =$ 2 min $(f(x) - g(x), 0)$ equals

(a) $f(x) + g(x) - |g(x) - f(x)|$
(b) $f(x) + g(x) \,|\, |g(x) - f(x)|$
(c) $f(x) - g(x) + |g(x) - f(x)|$
(d) $f(x) - g(x) - |g(x) - f(x)|$

3. The domain of the function $f(x) = ln\left(ln \frac{x}{\{x\}}\right)$ is (where {.} denotes the fractional part function)
(a) $(0, \infty) - I$ (b) $(1, \infty) - I$
(c) $R - I$ (d) $(2, \infty) - I$

4. If domain of f is D_1 and domain of g is D_2 then domain of $f + g$ is
(a) $D_1 \setminus D_2$
(b) $D_1 - (D_1 \setminus D_2)$
(c) $D_2 \setminus (D_2 \setminus D_1)$
(d) $D_1 \cap D_2$

5. $\sin ax + \cos ax$ and $|\sin x| + |\cos x|$ are periodic of same fundamental period if a equals
(a) 0 (b) 1
(c) 2 (d) 4

6. If $g(x)$ is a polynomial satisfying $g(x) g(y) = g(x) + g(y) + g(xy) - 2$ for all real x and y and $g(2) = 5$ then $g(3)$ is equal to
(a) 10 (b) 24
(c) 21 (d) None of these

7. The interval into which the function $y = \frac{(x-1)}{(x^2 - 3x + 3)}$ transforms the entire real line is
(a) $\left[\frac{1}{3}, 2\right]$
(b) $\left[-\frac{1}{3}, 1\right]$
(c) $\left[-\frac{1}{3}, 2\right]$
(d) None of these

8. Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined over the interval [0, 1]. The odd extensions of $f(x)$ to interval [–1, 1] is
(a) $x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(b) $-x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(c) $-x^2 + x + \sin x - \cos x + \log(1 + |x|)$
(d) None of these

9. The function $f(x) = \frac{\sec^{-1} x}{\sqrt{x - [x]}}$, where $[x]$ denotes the greatest integer less than or equal to x is defined for all x belonging to
(a) R
(b) $R - \{(-1, 1) \cup \{n : n \in I\}\}$
(c) $R^+ - (0, 1)$
(d) $R^+ - \{n : n \in N\}$

10. The period of the function
$f(x) = [\sin 3x] + |\cos 6x|$ is
([.] denotes the greatest integer less then or equal to x)
(a) π (b) $2\pi/3$
(c) 2π (d) $\pi/2$
(e) None of these

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (b)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (a)	**6.** (b)
7. (d)	**8.** (b)	**9.** (c)	**10.** (d)	**11.** (d)	**12.** (c)
13. (c)	**14.** (b)	**15.** (a)	**16.** (d)	**17.** (b)	**18.** (b)
19. (b)	**20.** (a)	**21.** (b)	**22.** (b)	**23.** (d)	**24.** (a)
25. (d)	**26.** (b)	**27.** (d)	**28.** (d)	**29.** (c)	**30.** (c)

31. (c)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (b)	**36.** (d)
37. (c)	**38.** (b)	**39.** (a)	**40.** (b)	**41.** (d)	**42.** (a)
43. (d)	**44.** (d)	**45.** (a)	**46.** (b)	**47.** (c)	**48.** (b)
49. (b)	**50.** (c)				

Multiple Choice –II

51. (c)	**52.** (b)	**53.** (a), (b), (c)	**54.** (d)	**55.** (d)	**56.** (b)
57. (d)	**58.** (a), (b), (c), (d)		**59.** (a), (d)	**60.** (a), (b), (c), (d)	
61. (b), (c), (d)	**62.** (b), (c)	**63.** (a)	**64.** (c)	**65.** (b)	**66.** (a), (c)
67. (a), (b)	**68.** (a), (d)	**69.** (c)	**70.** (d)	**71.** (d)	
72. (a)	**73.** (a), (b)	**74.** (c)	**75.** (d)		

Practice Test

1. (a)	**2.** (d)	**3.** (b)	**4.** (b), (c), (d)	**5.** (d)	**6.** (a)
7. (b)	**8.** (b)	**9.** (b)	**10.** (b)		

11

LIMITS

§ 11.1. Limit

Let $f(x)$ be a function of x. If for every positive number $\in$, however, small it may be, there exists a positive number δ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - l| < \in$ then we say, $f(x)$ *tends to limit* l *as* x *tends to* a and we write

$$\lim_{x \to a} f(x) = l$$

§ 11.2. Right hand and left hand Limits

In the definition of the limit we say that l is the limit of $f(x)$ *i.e.* $f(x) \to l$, when $x \to a$.

When $x \to a$, from the values of x greater than a, then the corresponding limit is **called the right hand limit (R.H.L.)** of $f(x)$ and is written as

$$\lim_{x \to a+} f(x) \quad \text{or} \quad f(a+0).$$

The working rule for finding the right hand limit is :

"Put $a + h$ for x in $f(x)$ and make h approach zero."

In short we have $\quad f(a+0) = \lim_{h \to 0} f(a+h)$

Similarly, when $x \to a$ from the values of x smaller (or less) than a, then the corresponding limit is called the **left-hand limit (L.H.L.)** of $f(x)$ and is written as

$$\lim_{x \to a-} f(x) \quad \text{or} \quad f(a-0)$$

The working rule for finding the left hand limit is :

"Put $a - h$ for x in $f(x)$ and make h approach zero."

In short we have

$$f(a-0) = \lim_{h \to 0} f(a-h)$$

If both these limits $f(a+0)$ and $f(a-0)$ **exist and are equal in value**, then their common value is the limit of the function $f(x)$ at $x = a$, *i.e.* l is the limit of $f(x)$ as $x \to a$ if

$$f(a+0) = l = f(a-0)$$

The limit of $f(x)$ as $x \to a$ **does not exist even if both these limits exist but are not equal in value** then also the limit of $f(x)$ as $x \to a$ does not exist.

§ 11.3. Frequently used Limits

(i) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e = \lim_{h \to 0} (1+h)^{1/h}$

(ii) $\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a$

(iii) $\lim_{h \to 0} (1 + ah)^{1/h} = e^a$

(iv) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ where $n \in Q$.

(v) $\lim_{\theta \to 0} \frac{\sin \theta^c}{\theta^c} = 1$ and $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180^\circ}$ $(180^\circ = \pi^c)$ and $\lim_{\theta \to 0} \frac{\tan \theta^c}{\theta^c} = 1$

(vi) $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \; (a > 0)$

(vii) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(viii) $\lim_{x \to \infty} \frac{\ln x}{x^m} = 0 \; (m > 0)$

(ix) $\lim_{x \to 0} \frac{(1 + x)^m - 1}{x} = m$

(x) $\lim_{x \to 0} \frac{\log_a (1 + x)}{x} = \log_a e \; (a > 0, a \neq 1)$

(xi) $\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} \{f(x) - 1\} g(x)}$

Provided $g(x) \to \infty$. $f(x) \to 1$ for $x \to a$.

(xii) $\lim_{n \to \infty} a^n = \begin{cases} \infty, \text{ (so no exits)} & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } -1 < a < 1 \\ \text{not exist,} & \text{if } a < -1 \end{cases}$

§ 11.4. Some Important Expansions

(i) $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots\ldots$

(ii) $\sin^{-1} x = x + \frac{1^2 . x^3}{3!} + \frac{1^2 . 3^2 . x^5}{5!} + \frac{1^2 . 3^2 . 5^2}{7!} x^7 + \ldots\ldots$

(iii) $(\sin^{-1} x)^2 = \frac{2}{2!} x^2 + \frac{2 . 2^2}{4!} x^4 + \frac{2 . 2^2 . 4^2}{6!} x^6 + \ldots\ldots$

(iv) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$

(v) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots$

(vi) $(1 + x)^{1/x} = e \left[1 - \frac{x}{2} + \frac{11}{24} x^2 + \ldots\ldots \right]$

§ 11.6. Indeterminate Forms

If a function $f(x)$ takes any of the following forms at $x = a$.

$$\frac{0}{0}, \; \frac{\infty}{\infty}, \; \infty - \infty, \; 0 \times \infty, \; 0^\circ, \; \infty^\circ, \; 1^\infty$$

then $f(x)$ is said to be indeterminate at $x = a$.

§ 11.7. L' Hospital's Rule

Let $f(x)$ and $g(x)$ be two functions, such that $f(a) = 0$ and $g(a) = 0$.

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$,

Provided $f'(a)$ and $g'(a)$ are not both zero.

Note : For other indeterminate terms we have to convert to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply *L'* Hospital's Rule.

Some times we have to repeat the process if the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of letters a, b, c, d whichever is appropriate :

1. If $0 < a < b$, then $\lim\limits_{n \to \infty} (b^n + a^n)^{1/n}$ is equal to

(a) e (b) a
(c) b (d) None of these

2. The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{1/2\,(1 - \cos 2x)}}{x}$ is

(a) 1 (b) –1
(c) 0 (d) none of these

3. The value of $\lim\limits_{x \to 0} \dfrac{ln\,(1 + \{x\})}{\{x\}}$ is where $\{x\}$ denotes the fractional part of x.

(a) 1 (b) 0
(b) 2 (d) does not exist

4. Let $f(x) = 1/\sqrt{(18 - x^2)}$, the value of $\lim\limits_{x \to 3} \left\{\dfrac{f(x) - f(3)}{x - 3}\right\}$ is

(a) 0 (b) –1/9
(c) –1/3 (d) 1/9

5. The value of $\lim\limits_{x \to \infty} \left(\dfrac{x^2 \sin(1/x) - x}{1 - |x|}\right)$ is

(a) 0 (b) 1
(c) –1 (d) None of these

6. $\lim\limits_{n \to \infty} \left(\dfrac{1}{n^3 + 1} + \dfrac{4}{n^3 + 1} + \dfrac{9}{n^3 + 1} + \ldots + \dfrac{n^2}{n^3 + 1}\right)$ is equal to

(a) 1 (b) 2/3
(c) 1/3 (d) 0

7. If $x > 0$ and g is bounded function, then $\lim\limits_{n \to \infty} \dfrac{f(x)\, e^{nx} + g(x)}{e^{nx} + 1}$ is

(a) $f(x)$ (b) $g(x)$
(c) 0 (d) None of these

8. The integer n for which $\lim\limits_{x \to 0} \dfrac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number is

(a) 1 (b) 2
(c) 3 (d) 4

9. $\lim\limits_{x \to 0} \dfrac{\sqrt{y^2 - (y - x)^2}}{(\sqrt{8xy - 4x^2} + \sqrt{8xy})^3}$ is equal to

(a) 1/4 (b) 1/2
(c) $1/2\sqrt{2}$ (d) None of these

10. $\lim\limits_{x \to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$ equal

(a) $-\pi$ (b) π
(c) $\pi/2$ (d) 1

11. $\lim\limits_{x \to \pi/2} \dfrac{\sin x - (\sin x)^{\sin x}}{1 - \sin x + ln \sin x}$ is equal to

(a) 4 (b) 2
(c) 1 (d) None of these

12. The value of the limit $\lim\limits_{x \to 0} \dfrac{a^{\sqrt{x}} - a^{1/\sqrt{x}}}{a^{\sqrt{x}} + a^{1/\sqrt{x}}}$, $a > 1$ is

(a) 4 (b) 2
(c) –1 (d) 0

13. $\lim\limits_{n \to \infty} \dfrac{n^{\alpha} \sin^2 n!}{n + 1}$, $0 < \alpha < 1$, is equal to

(a) 0 (b) 1
(c) ∞ (d) None of these

14. $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x - \tan^{-1} x}{x^2}$ is equal to

(a) 1/2 (b) –1/2
(c) 0 (d) ∞

15. $\lim\limits_{n \to \infty} \{\log_{(n-1)}(n) \log_n (n + 1) \log_{n+1} (n + 2) \ldots \log_{n^k - 1} (n^k)\}$ is equal to

(a) n (b) k
(c) ∞ (d) None of these

16. $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \cot^{-1} (r^2 + 3/4)$ is

(a) 0 (b) $\tan^{-1} 1$
(c) $\tan^{-1} 2$ (d) None of these

17. If $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x} + \dfrac{b}{x^2}\right)^{2x} = e^2$ then

(a) $a = 1, b = 2$ (b) $a = 2, b = 1$

(c) $a=1, b\in R$ (d) None of these

18. $\lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{e^{1/n}}{n}+\frac{e^{2/n}}{n}+\ldots+\frac{e^{(n-1)/n}}{n}\right)$ equals

(a) 0 (b) 1

(c) $e-1$ (d) None of these

19. $\lim\limits_{x\to\pi/4}\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}$ is equal to

(a) $\frac{3\sqrt{2}}{2}$ (b) $2\sqrt{2}$

(c) $\frac{4\sqrt{2}}{3}$ (d) does not exist

20. If $\lim\limits_{x\to 0}\frac{((a-n)\,nx-\tan x)\sin nx}{x^2}=0$, where n is non zero real number then a is equal to

(a) 0 (b) $\frac{n+1}{n}$

(c) n (d) $n+\frac{1}{n}$

21. If $\lim\limits_{x\to 0}\frac{(1+a^3)+8e^{1/x}}{1+(1-b^3)\,e^{1/x}}=2$ then

(a) $a=1, b=2$ (b) $a=1, b=-3^{1/3}$

(c) $a=2, b=3^{1/3}$ (d) None of these

22. If $\lim\limits_{x\to\infty}(\sqrt{(x^4-x^2+1)}-ax^2-b)=0$ then

(a) $a=1, b=-2$ (b) $a=1, b=1$

(c) $a=1, b=-1/2$ (d) None of these

23. If $S_n=\sum\limits_{k=1}^{n}a_k$ and $\lim\limits_{n\to\infty}a_n=a$, then $\lim\limits_{n\to\infty}\frac{S_{n+1}-S_n}{\sqrt{\sum\limits_{k=1}^{n}k}}$ is equal to

(a) 0 (b) a

(c) $\sqrt{2}\,a$ (d) $2a$

24. The value of $\lim\limits_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}$ is equal to

(a) $\frac{1}{5}$ (b) $\frac{1}{6}$

(c) $\frac{1}{4}$ (d) $\frac{1}{2}$

25. Let $f(x)=\begin{cases}x^2-1, & \text{if } 0<x<2\\ 2x+3, & \text{if } 2\le x<3\end{cases}$ the quadratic equation whose roots are $\lim\limits_{x\to 2-0}f(x)$ and $\lim\limits_{x\to 2+0}$ is

(a) $x^2-6x+9=0$ (b) $x^2-10x+21=0$

(c) $x^2-14x+49=0$ (d) None of these

26. $\lim\limits_{x\to 1}\frac{x\sin\{x-[x]\}}{x-1}$, where [.] denotes the greatest integer function, is

(a) 0 (b) –1

(c) not existent (d) None of these

27. The value of $\lim\limits_{x\to 0}[x^2+x+\sin x]$ where [.] denotes the greatest integer function

(a) does not exist (b) is equal to zero

(c) –1 (d) None of these

28. $\lim\limits_{x\to\pi/4}(2-\tan x)^{1/\ln(\tan x)}$ equals to

(a) e (b) 1

(c) 0 (d) e^{-1}

29. $\lim\limits_{n\to\infty}\left(\frac{n!}{(mn)^n}\right)^{1/n}$ $(m\in N)$ is equal to

(a) $1/em$ (b) m/e

(c) em (d) e/m

30. If $y=2^{-2^{1/(1-x)}}$ then $\lim\limits_{x\to 1+}y$ is

(a) –1 (b) 1

(c) 0 (d) 1/2

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

31. Let $f(x)=\lim\limits_{n\to\infty}\frac{x^{2n}-1}{x^{2n}+1}$, then

(a) $f(x)=1$ for $|x|>1$

(b) $f(x)=-1$ for $|x|<1$

(c) $f(x)$ is not defined for any value of x

(d) $f(x)=1$ for $|x|=1$

32. The graph of the function $y = f(x)$ has a unique tangent at the point $(a, 0)$ through which the graph passes. Then $\lim_{x \to a} \frac{\log_e \{1 + 6f(x)\}}{3f(x)}$ is

(a) 0 (b) 1
(c) 2 (d) None of these

33. $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$

(a) does not exist (b) is equal to $-3/2$
(c) is equal to $3/2$ (d) is equal to 3

34. If $\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1$ then

(a) $a = -5/2, b = -1/2$
(b) $a = -3/2, b = -1/2$
(c) $a = -3/2, b = -5/2$
(d) $a = -5/2, b = -3/2$

35. $\lim_{x \to 0} \frac{x^n \sin^n x}{x^n - \sin^n x}$ is non zero finite then n must be equal to

(a) 1 (b) 2
(c) 3 (d) None of these

36. $\lim_{x \to \infty} (\sqrt{(x^2 + x)} - x)$ equals

(a) $\lim_{x \to 0} \frac{x + \ln(1 - x)}{x^2}$
(b) $\lim_{x \to 0} \frac{e^{-x} - 1 + x}{x^2}$
(c) $\lim_{x \to 0} \frac{-\sqrt{x}}{\sqrt{x} + \sqrt{x^2 + 2x}}$
(d) $\lim_{x \to 0} \frac{\cos x^2 - 1}{x^4}$

37. If $[x]$ denotes the greatest integer less than or equal to x, then the value of $\lim_{x \to 1} (1 - x + [x - 1] + |1 - x|)$ is

(a) 0 (b) 1
(c) -1 (d) None of these

38. $\lim_{x \to 0} \frac{1}{x} \left(\int_y^c e^{\sin^2 t}\, dt - \int_{x+y}^c e^{\sin^2 t}\, dt \right)$ is equal to (where c is a constant)

(a) $e^{\sin^2 y}$ (b) $\sin 2y\, e^{\sin^2 y}$
(c) 0 (d) None of these

39. If α and β be the roots of $ax^2 + bx + c = 0$, then $\lim_{x \to \alpha} (1 + ax^2 + bx + c)^{1/(x - \alpha)}$ is

(a) $a(\alpha - \beta)$ (b) $\ln |a(\alpha - \beta)|$
(c) $e^{a(\alpha - \beta)}$ (d) $e^{a|\alpha - \beta|}$

40. If $f(x) = \begin{cases} \sin x, & x \neq n\pi,\ n \in I \\ 2, & \text{otherwise} \end{cases}$ and

$g(x) = \begin{cases} x^2 + 1, & x \neq 0, 2 \\ 4, & x = 0 \\ 5, & x = 2 \end{cases}$ then $\lim_{x \to 0} g\{f(x)\}$ is

(a) 5 (b) 6
(c) 7 (d) 1

41. If $A_i = \frac{x - a_i}{|x - a_i|}$, $i = 1, 2, 3, \ldots, n$ and if $a_1 < a_2 < a_3 \ldots < a_n$. Then $\lim_{x \to a_m} (A_1 A_2 \ldots A_n)$, $1 \le m \le n$

(a) is equal to $(-1)^m$
(b) is equal to $(-1)^{m+1}$
(c) is equal to $(-1)^{m-1}$
(d) does not exist

42. $\lim_{x \to 1+0} \frac{\int_1^x |t - 1|\, dt}{\sin(x - 1)}$ is equal to

(a) 0 (b) 1
(c) -1 (d) None of these

43. $\lim_{x \to \infty} \frac{\log_e [x]}{x}$, where [.] denotes the greatest integer function is

(a) 0 (b) 1
(c) -1 (d) non existent

44. If $[x]$ denotes the greatest integer $< x$, then $\lim_{n \to \infty} \frac{1}{n^3} \{[1^2 x] + [2^2 x] + [3^2 x] + \ldots + [n^2 x]\}$ equal

(a) $x/2$ (b) $x/3$
(c) $x/6$ (d) 0

45. The value of the limit

$\lim_{x \to 0} \{1^{1/\sin^2 x} + 2^{1/\sin^2 x} + \ldots + n^{1/\sin^2 x}\}^{\sin^2 x}$

(a) ∞ (b) 0
(c) $\frac{n(n+1)}{2}$ (d) n

Practice Test

M.M. : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. $\lim_{x \to 0} \dfrac{x^n - \sin x^n}{x - \sin^n x}$ is non zero finite then n must be equal to

(a) 1 (b) 2

(c) 3 (d) None of these

2. $\lim_{x \to 0} \dfrac{\log x^n - [x]}{[x]}$, $n \in N$, ($[x]$ denotes greatest integer less than or equal to x)

(a) has value –1 (b) has value 0

(c) has value 1 (d) Does not exist

3. $\lim_{x \to 0} \dfrac{\sin [\cos x]}{1 + [\cos x]}$ ([.] denotes the gratest integer function)

(a) equal to 1 (b) equal to 0

(c) Does not exist (d) None of these

4. If α and β be the roots of $ax^2 + bx + c = 0$, then

$$\lim_{x \to \alpha} \frac{1 - \cos (ax^2 + bx + c)}{(x - \alpha)^2} \text{ is equal to}$$

(a) 0 (b) $\frac{1}{2}(\alpha - \beta)^2$

(c) $\frac{a^2}{2}(\alpha - \beta)^2$ (d) $-\frac{a^2}{2}(\alpha - \beta)^2$

5. If x is a real number is [0, 1], then the value of

$$f(x) = \lim_{m \to \infty} \lim_{n \to \infty} \{1 + \cos^{2m}(n!\, \pi x)\}$$

is given by

(a) 2 or 1 according as x is rational or irrational

(b) 1 or 2 according as x is rational or irrational

(c) 1 for all x

(d) 2 or 1 for all x

6. The value of $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e + \frac{1}{2}ex}{x^2}$ is

(a) $\frac{11}{24}e$ (b) $-\frac{11e}{24}$

(c) $\frac{e}{24}$ (d) None of these

7. $\lim_{x \to \infty} \sqrt{x}\,(\sqrt{x + 1} - \sqrt{x})$ equals

(a) $\lim_{x \to 0} \dfrac{\ln (1 + x) - x}{x^2}$

(b) $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$

(c) $\lim_{x \to 0} \dfrac{\sqrt{(1 + x)} - 1}{x}$

(d) $\lim_{x \to 0} \dfrac{\sqrt{x}}{\sqrt{x + \sqrt{(x^2 + 2x)}}}$

8. $\lim_{x \to 0} \dfrac{\tan ([-\pi^2]\, x^2) - \tan ([-\pi^2])\, x^2}{\sin^2 x}$ equals where [.] denotes the greater integer function

(a) 0 (b) 1

(c) tan 10–10 (d) ∞

9. Let $a = \min \{x^2 + 2x + 3, x \in R\}$ and $b = \lim_{\theta \to 0} \dfrac{1 - \cos \theta}{\theta^2}$. The value of $\sum_{r=0}^{n} a^r . b^{n-r}$ is

(a) $\dfrac{2^{n+1} - 1}{3.2^n}$ (b) $\dfrac{2^{n+1} + 1}{3.2^n}$

(c) $\dfrac{4^{n+1} - 1}{3.2^n}$ (d) None of these

10. $\lim_{x \to 0} \left(\dfrac{\sin x}{x}\right)^{\left(\frac{\sin x}{x - \sin x}\right)}$ equals

(a) 1 (b) e

(c) e^{-1} (d) e^{-2}

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (b)	**2.** (d)	**3.** (d)	**4.** (d)	**5.** (a)	**6.** (c)
7. (a)	**8.** (c)	**9.** (d)	**10.** (b)	**11.** (b)	**12.** (c)
13. (a)	**14.** (c)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (c)
19. (a)	**20.** (d)	**21.** (b)	**22.** (c)	**23.** (a)	**24.** (b)
25. (b)	**26.** (c)	**27.** (a)	**28.** (d)	**29.** (a)	**30.** (b)

Multiple Choice –II

31. (a), (b), (c)	**32.** (c)	**33.** (d)	**34.** (d)	**35.** (b)	**36.** (b)
37. (d)	**38.** (c)	**39.** (c)	**40.** (d)	**41.** (d)	**42.** (a)
43. (a)	**44.** (b)	**45.** (d)			

Practice Test

1. (a)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (a)
7. (b), (c)	**8.** (c)	**9.** (c)	**10.** (c)		

12

CONTINUITY AND DIFFERENTIABILITY

§ 12.1. Continuity of a Function

Continuity of a function $f(x)$ can be discussed in two ways *(1) at a point (2) in an interval.*

(1) The function $f(x)$ is said to be continuous at the point $x = a$ if

$$\lim_{x \to a} f(x) \text{ exist } = f(a)$$

i.e.
$$\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = [f(x)]_{x = a}$$

or L.H.L. = R.H.L. = value of function at $x = a$.

(2) The function $f(x)$ is said to be continuous in an interval $[a, b]$, if $f(x)$ be continuous at every point of the interval.

In other words, the function $f(x)$ is continuous in interval $[a, b]$. It is exist (be not indeterminate) and be finite ($\neq \infty$) for all values of x in interval $[a, b]$.

§ 12.2. Discontinuity of a Function

The discontinuity of a function $f(x)$ at $x = a$ can arise in two ways

(1) If $\lim_{x \to a-} f(x)$ exist but $\neq f(a)$ or $\lim_{x \to a+} f(x)$ exist but $\neq f(a)$, then the function $f(x)$ is said to have a ***removable discontinuity.***

(2) The function $f(x)$ is said to have an ***unremovable discontinuity*** when $\lim_{x \to a} f(x)$ does not exist.

i.e.,
$$\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x).$$

§ 12.3. Right hand and Left hand Derivatives of a Function

The progressive derivative or right hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad h > 0$$

if it exists finitely and is denoted by $Rf'(a)$ or by $f'(a+)$ or by $f_+'(a)$.

The regressive or left hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}, \quad h > 0$$

if it exists finitely and denoted by $Lf'(a)$ or by $f'(a-)$ or by $f_-'(a)$.

§ 12.4. Derivability or Differentiability of a Function

The function $f(x)$ is said to be ***differentiable*** at $x = a$ if $Rf'(a)$ and $Lf'(a)$ both exist finitely and are equal, and their common value is called the ***derivative or differential coefficient*** at the point $x = a$.

The function $f(x)$ is said to be non differentiable at $x = a$ if

(i) both $Rf'(a)$ and $Lf'(a)$ exist but are not equal (ii) either or both $Rf'(a)$ and $Lf'(a)$ are not finite

(iii) either or both $Rf'(a)$ and $Lf'(a)$ do not exist.

§ 12.5. Differentiability and Continuity

(i) If $Rf'(a)$ and $Lf'(a)$ exist finitely (both may or may not be equal) then $f(x)$ is continuous at $x = a$

(ii) If $f(x)$ is differentiable at every point of its domain, then it must be continuous in that domain.

(iii) The converse of the above result (ii) is not true. *i.e.*, If $f(x)$ is continuous at $x = a$ then it may or may not be differentiable at $x = a$.

(iv) If $f(x)$ is differentiable then its graph must be smooth *i.e.*, there should be no break or corner.

(v) For a function $f(x)$:

(a) Differentiable $\Rightarrow$ Continuous (from (ii))

(b) Continuous $\not\Rightarrow$ Differentiable (from (iii))

(c) Not continuous $\Rightarrow$ Not differentiable

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letter a, b, c, d whichever is appropriate :

1. The value of the $f(0)$, so that the function
$$f(x)=\frac{\sqrt{(a^2-ax+x^2)}-\sqrt{(a^2+ax+x^2)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$$
becomes continuous for all x, is given by

(a) $a\sqrt{a}$ (b) $\sqrt{a}$

(c) $-\sqrt{a}$ (d) $-a\sqrt{a}$

2. Let $f(x)=\tan(\pi/4-x)/\cot 2x$ $(x\neq\pi/4)$. The value which should be assigned to f at $x=\pi/4$. so that it is continuous every where, is

(a) 1/2 (b) 1

(c) 2 (d) None of these

3. If $f(x)=\dfrac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}$ $(x\neq 0)$: then for f to be continuous every where, $f(0)$ is equal to

(a) –1 (b) 1

(c) 2^6 (d) None of these

4. Let $f''(x)$ be continuous at $x=0$ and $f''(0)=4$ the value of
$$\lim_{x\to 0}\frac{2f(x)-3f(2x)+f(4x)}{x^2}$$ is

(a) 11 (b) 2

(c) 12 (d) None of these

5. Let f be a function satisfying $f(x+y)=f(x)+f(y)$ and $f(x)=x^2 g(x)$ for all x and y, where $g(x)$ is a continuous function then $f'(x)$ is equal to

(a) $g'(x)$ (b) $g(0)$

(c) $g(0)+g'(x)$ (d) 0

6. Let $f(x+y)=f(x)f(y)$ for all x and y. Suppose that $f(3)=3$ and $f'(0)=11$ then $f'(3)$ is given by

(a) 22 (b) 44

(c) 28 (d) None of these

7. Let $f: R\to R$ be a differentiable function and $f(1)=4$. Then the value of $\lim_{x\to 1}\int_4^{f(x)}\frac{2t}{x-1}\,dt$ is

(a) $8f'(1)$ (b) $4f'(1)$

(c) $2f'(1)$ (d) $f'(1)$

8. Let $f: R\to R$ be a function such that $f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{3}$, $f(0)=3$ and $f'(0)=3$, then

(a) $\dfrac{f(x)}{x}$ is differentiable in R

(b) $f(x)$ is continuous but not differentiable in R

(c) $f(x)$ is continuous in R

(d) $f(x)$ is bounded in R

9. If $f(x)=\dfrac{\tan\pi\,[(2\pi-3\pi)^3]}{1+[2x-3\pi]^2}$ ([.] denotes the greatest integer function), then

(a) $f(x)$ is continuous in R

(b) $f(x)$ is continuous in R but not differentiable in R

(c) $f'(x)$ exists everywhere but $f''(x)$ does not exist at some $x\in R$

(d) None of these

10. If $f(x)=\begin{cases}1, & x \text{ is rational}\\ 2, & x \text{ is irrational}\end{cases}$ then

(a) $f(x)$ is continuous in $R \sim I$

(b) $f(x)$ is continuous in $R \sim Q$
(c) $f(x)$ is continuous in R but not differentiable in R
(d) $f(x)$ is neither continuous not differentiable in R

11. If $f(x)$ is a twice differentiable function, then between two consecutive roots of the equation $f'(x) = 0$ there exists
(a) at least one root of $f(x) = 0$
(b) at most one root of $f(x) = 0$
(c) exactly one root of $f(x) = 0$
(d) at most one root of $f''(x) = 0$

12. If the derivative of the function $f(x) = \begin{cases} bx^2 + ax + 4 \;;\; x > -1 \\ ax^2 + b \;;\; x < -1 \end{cases}$ is everywhere continuous, then
(a) $a = 2, b = 3$ (b) $a = 3, b = 2$
(c) $a = -2, b = -3$ (d) $a = -3, b = -2$

13. If $f(x) = \begin{cases} \sin x,\, x \neq n\pi,\; n \in I \\ 2,\;\text{otherwise} \end{cases}$ and $g(x) = \begin{cases} x^2 + 1,\; x \neq 0, 2 \\ 4\;,\; x = 0 \\ 5\;,\; x = 2 \end{cases}$ then $\lim_{x \to 0} g\{f(x)\}$ is
(a) 5 (b) 6
(c) 7 (d) 1

14. The function $f(x) = |2\, Sgn\, 2x| + 2$ has
(a) jump discontinuity
(b) removal discontinuity
(c) infinite discontinuity
(d) no discontinuity at $x = 0$

15. If the function
$$f(x) = \begin{cases} (1 + |\sin x|)^{a/|\sin x|}, & -\frac{\pi}{6} < x < 0 \\ b, & x = 0 \\ e^{\tan 2x/\tan 3x}, & 0 < x < \frac{\pi}{6} \end{cases}$$
is continuous at $x = 0$ then
(a) $a = \log_e b,\, a = 2/3$
(b) $b = \log_e a,\, a = 2/3$
(c) $a = \log_e b,\, b = 2$
(d) None of these

16. If $f(x) = \int_{-1}^{x} |t|\, dt,\; x \geq -1$, then
(a) f and f' are continuous for $x + 1 > 0$
(b) f is continuous but f' is not so for $x + 1 > 0$
(c) f and f' are continuous at $x = 0$
(d) f is continuous at $x = 0$ but f' is not so.

17. Let $f(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right);\; x \neq 0 \\ 0\;;\; x = 0 \end{cases}$ then $f(x)$ is continuous but not differentiable at $x = 0$ if
(a) $n \in (0, 1]$ (b) $n \in [1, \infty)$
(c) $n \in (-\infty, 0)$ (d) $n = 0$

18. If $f(x) = \dfrac{\sqrt{(a^2 - ax + x^2)} - \sqrt{(a^2 + ax + x^2)}}{\sqrt{(a + x)} - \sqrt{(a - x)}}$, $x \neq 0$ then the value of $f(0)$ such that $f(x)$ is continuous at $x = 0$ is
(a) $\dfrac{a\sqrt{a}}{|a|}$ (b) $-\dfrac{a\sqrt{a}}{|a|}$
(c) $\dfrac{\sqrt{a}}{|a|}$ (d) $-\dfrac{\sqrt{a}}{|a|}$

19. Let [.] represent the greatest integer function and $f(x) = [\tan^2 x]$ then
(a) $\lim_{x \to 0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is non-differentiable at $x = 0$
(d) $f'(0) = 1$

20. $f(x) = \begin{cases} x - [x], & \text{for } 2n \leq x < 2n + 1,\; n \in N \\ & \text{where } [x] = \text{Integral part of } x < x \\ \frac{1}{2}, & \text{for } 2n + 1 < x < 2n + 2 \end{cases}$
the function
(a) is discontinuous at $x = 1, 2$
(b) is periodic with period 1
(c) is periodic with period 2
(d) $\int_0^2 f(x)\, dx$ exists

21. A function
$$f(x) = x\left[1 + \frac{1}{3}\sin(\log x^2)\right], x \neq 0$$
$$f(0) = 0$$
[.] = Integral part, The function
(a) is continuous at $x = 0$
(b) is monotonic
(c) is derivable at $x = 0$
(d) con not be defined for $x < -1$

22. If $f(x) = \begin{cases} [\cos \pi x], & x < 1 \\ |x - 1|, & 1 < x < 2 \end{cases}$

([.] denotes the greatest integer function) then $f(x)$ is
(a) Continuous and non-differentiable at $x = -1$ and $x = 1$
(b) Continuous and differentiable at $x = 0$
(c) Discontinuous at $x = 1/2$
(d) Continuous but not differentiable at $x = 2$

23. If $f(x) = [\sqrt{2}\sin x]$, where $[x]$ represents the greatest integer function $\le x$, then
(a) $f(x)$ is periodic
(b) Maximum value of $f(x)$ is 1 in the interval $[-2\pi, 2\pi]$
(c) $f(x)$ is discontinuous at $x = \frac{n\pi}{2} + \frac{\pi}{4}, n \in I$
(d) $f(x)$ is differentiable at $x = n\pi,\ n \in I$

24. The function defined by $f(x) = (-1)^{[x^3]}$ ([.] denotes greatest integer function) satisfies
(a) Discontinuous for $x = n^{1/3}$, where n is any integer
(b) $f(3/2) = 1$ (c) $f'(x) = 0$ for $-1 < x < 1$
(d) None of these

25. If $f(x)$ be a continuous function defined for $1 \le x \le 3$ $f(x) \in Q\ \forall\ x \in [1, 3]$, $f(2) = 10$, then $f(1{\cdot}8)$ is
(where Q is a set of all rational numbers)
(a) 1 (b) 5
(c) 10 (d) 20

26. Let $f(x) = \begin{cases} \left| x^p \sin\left(\frac{1}{x}\right) + x\,|x| \right|, & x \ne 0 \\ 0, & x = 0 \end{cases}$

the set of values, of p for which $f''(x)$ is everywhere continuous is
(a) $[2, \infty)$ (b) $[-3, \infty)$
(c) $[5, \infty)$ (d) None of these

27. The value of p for which the function
$$f(x) = \frac{(4^x - 1)^3}{\sin(x/p)\,ln\left(1 + \frac{x^2}{3}\right)},\ x \ne 0$$
$$= 12\,(ln\,4)^3,\ x = 0$$
may be continuous at $x = 0$ is
(a) 1 (b) 2
(c) 3 (d) 4

28. The value of $f(0)$ so that the function $f(x) = \frac{1 - \cos(1 - \cos x)}{x^4}$ is continuous everywhere is
(a) 1/8 (b) 1/2
(c) 1/4 (d) None of these

29. The jump of the function at the point of the discontinuity of the function
$$f(x) = \frac{1 - k^{1/x}}{1 + k^{1/x}}\ (k > 0) \text{ is}$$
(a) 4 (b) 2
(c) 3 (d) None of these

30. Let $f''(x)$ be continuous at $x = 0$ and $f''(0) = 4$. The value of $\lim_{x \to 0} \frac{2f(x) - 3f(2x) + f(4x)}{x^2}$ is
(a) 6 (b) 10
(c) 11 (d) 12

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer (s).

31. Let $f(x) = \frac{1}{[\sin x]}$ ([.] denotes the greatest integer function) then
(a) Domain of $f(x)$ is $(2n\pi + \pi, 2n\pi + 2\pi) \cup \{2n\pi + \pi/2\}$, where $n \in I$
(b) $f(x)$ is continuous when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$
(c) $f(x)$ is differentiable at $x = \pi/2$
(d) None of these

32. Let $g(t) = [t(1/t)]$ for $t > 0$ ([.] denotes the greatest integer function), then $g(g)$ has
(a) Discontinuities at finite number of points
(b) Discontinuities at infinite number of points
(c) $g(1/2) = 1$ (d) $g(3/4) = 1$

33. Let $f(x) = [x] + \sqrt{x - [x]}$, where $[x]$ denotes the greatest integer function. Then
(a) $f(x)$ is continuous on R^+
(b) $f(x)$ is continuous on R

(c) $f(x)$ is continuous on $R \sim I$

(d) None of these

34. Let $f(x)$ and $\phi(x)$ be defined by $f(x) = [x]$ and $\phi(x) = \begin{cases} 0, & x \in I \\ x^2, & x \in R - I \end{cases}$ (where [.] denotes the greatest integer function) then

(a) $\lim_{x \to 1} \phi(x)$ exists, but ϕ is not continuous at $x = 1$

(b) $\lim_{x \to 1} f(x)$ does not exist and f is not continuous at $x = 1$

(c) $\phi\, o f$ is continuous for all x

(d) $f o \phi$ is continuous for all x

35. The following functions are continuous on $(0, \pi)$

(a) $\tan x$

(b) $\int_0^{\pi} t \sin\left(\frac{1}{t}\right) dt$

(c) $\begin{cases} 1, & 0 < x \le \frac{3\pi}{4} \\ 2 \sin \frac{2}{9} x, & \frac{3\pi}{4} < x < \pi \end{cases}$

(d) $\begin{cases} x \sin x, & 0 < x \le \pi/2 \\ \frac{\pi}{2} \sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

36. If $f(x) = \dfrac{\tan [x] \pi}{[1 + | ln (\sin^2 x + 1) |]}$, where [.] denotes the greatest integer function, then $f(x)$ is

(a) continuous $\forall\, x \in R$

(b) discontinuous $\forall\, x \in I$

(c) non-differentiable $\forall\, x \in I$

(d) a periodic function with fundamental period not defined

37. If $f'(x) = g(x)(x - a)^2$ where $g(a) \ne 0$ and g is continuous at $x = a$ then

(a) f is increasing near a if $g(a) > 0$

(b) f is increasing near a if $g(a) < 0$

(c) f is decreasing near a if $g(a) > 0$

(d) f is decreasing near a if $g(a) < 0$

38. A function which is continuous and not differentiable at the origin is

(a) $f(x) = x$ for $x < 0$ and $f(x) = x^2$ for $x \ge 0$

(b) $g(x) = x$ for $x < 0$ and $g(x) = 2x$ for $x \ge 0$

(c) $h(x) = x\,|x|$, $x \in R$

(d) $k(x) = 1 + |x|$, $x \in R$

39. Let $f(x) = \lim_{n \to \infty} (\sin x)^{2n}$, then f is

(a) Continuous at $x = \pi/2$

(b) Discontinuous at $x = \pi/2$

(c) Discontinuous at $x = -\pi/2$

(d) Discontinuous at an infinite number of points

40. If $f(x) = \tan^{-1} \cot x$, then

(a) $f(x)$ is periodic with period π

(b) $f(x)$ is discontinuous at $x = \pi/2, 3\pi/2$

(c) $f(x)$ is not differentiable at $x = \pi, 99\pi, 100\pi$

(d) $f(x) = -1$, for $2n\pi < x < (2n + 1)\pi$

41. Let $f(x) = \begin{cases} \int_0^x (1 + |1 - t|)\, dt, & x > 2 \\ 5x + 1, & x < 2 \end{cases}$

Then

(a) $f(x)$ is not continuous at $x = 2$

(b) $f(x)$ is continuous but not differentiable at $x = 2$

(c) $f(x)$ is differentiable everywhere

(d) The right derivative of $f(x)$ at $x = 2$ does not exist

42. $f(x) = \min\{1, \cos x, 1 - \sin x\}$, $-\pi \le x \le \pi$ then

(a) $f(x)$ is not differentiable at '0'

(b) $f(x)$ is differentiable at $\pi/2$

(c) $f(x)$ has local maxima at '0'

(d) None of these

43. If $f(x) = \begin{cases} \dfrac{x \log \cos x}{\log(1 + x^2)}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ then

(a) f is continuous at $x = 0$

(b) f is continuous at $x = 0$ but not differentiable at $x = 0$

(c) f is differentiable at $x = 0$

(d) f is not continuous at $x = 0$

44. Let $f(x) = \dfrac{\sin(\pi[x - \pi])}{1 + [x^2]}$ where [.] denotes the greatest integer function. Then $f(x)$ is

(a) Continuous at integral points

(b) Continuous everywhere but not differentiable

(c) Differentiable once but $f'(x), f''(x), \ldots$ do not exist

(d) Differentiable for all x

45. The function $f(x) = \begin{cases} |2x-3|[x] ; & x > 1 \\ \sin\left(\dfrac{\pi x}{2}\right); & x < 1 \end{cases}$

([.] denotes the greatest integer function)

(a) is continuous at $x = 0$
(b) is differentiable at $x = 0$
(c) is continuous but not differentiable at $x = 1$
(d) is continuous but not differentiable at $x = 3/2$

46. Let $f(x) = \begin{cases} \dfrac{a|x^2-x-2|}{2+x-x^2}, & x < 2 \\ b, & x = 2 \\ \dfrac{x-[x]}{x-2}, & x > 2 \end{cases}$

([.] denotes the greatest integer function)
If $f(x)$ is continuous at $x = 2$ then

(a) $a = 1, b = 2$ (b) $a = 1, b = 1$
(c) $a = 0, b = 1$ (d) $a = 2, b = 1$

47. The function $f(x) = \dfrac{e^{\tan x} - 1}{e^{\tan x} + 1}$ is discontinuous at $x =$

(a) $n\pi + \pi$ (b) $n\pi + \pi/2$
(c) $n\pi + \pi/4$ (d) $n\pi + \pi/8$

48. Let $f(x) = \begin{cases} -1, & x \le 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ and $g(x) = \sin x + \cos x$, then points of discontinuity of $f\{g(x)\}$ in $(0, 2\pi)$ is

(a) $\left\{\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right\}$ (b) $\left\{\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right\}$
(c) $\left\{\dfrac{2\pi}{3}, \dfrac{5\pi}{3}\right\}$ (d) $\left\{\dfrac{5\pi}{4}, \dfrac{7\pi}{3}\right\}$

49. The points of discontinuity of the function

$f(x) = \lim\limits_{n \to \infty} \dfrac{(2\sin x)^{2n}}{3^n - (2\cos x)^{2n}}$ are given by

(a) $n\pi \pm \pi/12$ (b) $n\pi \pm \pi/6$
(c) $n\pi \pm \pi/3$ (d) None of these

50. Let $f(x) = [\cos x + \sin x], 0 < x < 2\pi$ where $[x]$ denotes the greatest integer less than or equal to x. The number of points of discontinuity of $f(x)$ is

(a) 6 (b) 5
(c) 4 (d) 3

51. The function $f(x) = |x^2 - 3x + 2| + \cos|x|$ is not differentiable at $x =$

(a) -1 (b) 0
(c) 1 (d) 2

52. Let $h(x) = \min\{x, x^2\}$ for every real number x. Then

(a) h is continuous for all x
(b) h is differentiable for all x
(c) $h'(x) = 1$ for all $x > 1$
(d) h is not differentiable at two values of x

53. Let $f(x) = \dfrac{1 - \tan x}{4x - \pi}, x \ne \pi/4$ and $x \in [0, \pi/2)$
$= \lambda, x = \pi/4$
If $f(x)$ is continuous in $(0, \pi/2]$ then λ is

(a) 1 (b) 1/2
(c) $-1/2$ (d) None of these

54. A function $f(x)$ is defined in the interval $[1, 4]$ as follows :

$f(x) = \begin{cases} \log_e[x], & 1 < x < 3 \\ |\log_e x|, & 3 \le x < 4 \end{cases}$

the graph of the function $f(x)$

(a) is broken at two points
(b) is broken at exactly one point
(c) does not have a definite tangent at two points
(d) does not have a definite tangent at more than two points

55. If $g(x) = \begin{cases} [f(x)], & x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right) \\ 3, & x = \pi/2 \end{cases}$

where $[x]$ denotes the greatest integer function and

$f(x) = \dfrac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|}, n \in R$

then

(a) $g(x)$ is continuous and differentiable at $x = \pi/2$ when $0 < x < 1$
(b) $g(x)$ is continuous and differentiable at $x = \pi/2$ when $n > 1$
(c) $g(x)$ is continuous but not differentiable at $x = \pi/2$ when $0 < n < 1$
(d) $g(x)$ is continuous but not differentiable, at $x = \pi/2$ when $n > 1$

Practice Test

M.M. : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. $f(x) = 1 + x(\sin x)[\cos x]$, $0 < x \le \pi/2$

([.] denotes the greatest integer function)

(a) is continuous in $(0, \pi/2)$

(b) is strictly decreasing in $(0, \pi/2)$

(c) is strictly increasing in $(0, \pi/2)$

(d) has global maximum value 2

2. If $f(x) = \min(\tan x, \cot x)$ then

(a) $f(x)$ is discontinuous at $x = 0, \pi/4, 5\pi/4$

(b) $f(x)$ is continuous at $x = 0, \pi/2, 3\pi/2$

(c) $\int_0^{\pi/2} f(x)\, dx = 2 \ln \sqrt{2}$

(d) $f(x)$ is periodic with period π

3. If $f(x)$ is a continuous function $\forall x \in R$ and the range of $f(x)$ is $(2, \sqrt{26})$ and $g(x) = \left[\frac{f(x)}{c}\right]$ is continuous $\forall x \in R$, then the least positive integral value of c is, where [.] denotes the greatest integer function

(a) 2 (b) 3

(c) 5 (d) 6

4. $f(x) = \frac{[1/2 + x] - [1/2]}{x}$, $-1 < x < 2$

has ([.] denotes the greatest integer function).

(a) Discontinuity at $x = 0$

(b) Discontinuity at $x = 1/2$

(c) Discontinuity at $x = 1$

(d) Discontinuity at $x = 3/2$

5. If $f(x) = \begin{cases} a + \frac{\sin [x]}{x}, & x > 0 \\ 2, & x = 0 \\ b + \left[\frac{\sin x - x}{x^3}\right], & x < 0 \end{cases}$

(where [.] denotes the greatest integer function). If $f(x)$ is continuous at $x = 0$ then b is equal to

(a) $a - 2$ (b) $a - 1$

(c) $a + 1$ (d) $a + 2$

6. Which of the following functions are differentiable in $(-1, 2)$

(a) $\int_x^{2x} (\log x)^2\, dx$ (b) $\int_x^{2x} \frac{\sin x}{x}\, dx$

(c) $\int_0^x \frac{1 - t + t^2}{1 + t + t^2}\, dt$ (d) None of these

7. If $f(x) = [x] + [x + 1/3] + [x + 2/3]$, then ([.] denotes the greatest integer function)

(a) $f(x)$ is discontinuous at $x = 1, 10, 15$

(b) $f(x)$ is continuous at $x = n/3$, where n is any integer

(c) $\int_0^{2/3} f(x)\, dx = 1/3$

(d) $\lim_{x \to 2/3} f(x) = 2$

8. Let

$$f(x) = \begin{cases} (1 + |\cos x|)^{b/|\cos x|}, & n\pi < x < (2n+1)\pi/2 \\ e^a \cdot e^b, & x = (2n+1)\pi/2 \\ e^{\cot 2x / \cot 8x}, & (2n+1)\pi/2 < x < (n+1)\pi \end{cases}$$

If $f(x)$ is continuous in $(n\pi, (n+1)\pi)$ then

(a) $a = 1, b = 2$ (b) $a = 2, b = 2$

(c) $a = 2, b = 3$ (d) $a = 3, b = 4$

9. If $f(x) = \begin{cases} (\sin^{-1} x)^2 \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ then

(a) $f(x)$ is continuous everywhere in $x \in [-1, 1]$

(b) $f(x)$ is continuous no where in $x \in [-1, 1]$

(c) $f(x)$ is differentiable everywhere in $x \in (-1, 1)$

(d) $f(x)$ is differentiable no where in $x \in [-1, 1]$

10. Let $f(x) = [\alpha + \beta \sin x]$, $x \in (0, \pi)$, $\alpha \in I$, β is a prime number and $[x]$ is the greatest integer less than or equal to x. The number of points at which $f(x)$ is not differentiable is

(a) β (b) $\beta - 1$

(c) $2\beta + 1$ (d) $2\beta - 1$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answer

Multiple Choice –I

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (d)	**6.** (d)
7. (a)	**8.** (c)	**9.** (a)	**10.** (d)	**11.** (b)	**12.** (a)
13. (d)	**14.** (b)	**15.** (a)	**16.** (a)	**17.** (a)	**18.** (b)
19. (b)	**20.** (b)	**21.** (a)	**22.** (c)	**23.** (c)	**24.** (a)
25. (c)	**26.** (c)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (d)

Multiple Choice –II

31. (a), (b)	**32.** (b), (c)	**33.** (b)	**34.** (a), (b), (c)	**35.** (b), (c)	**36.** (a), (d)
37. (a), (d)	**38.** (a), (b), (d)	**39.** (b), (c), (d)	**40.** (a), (c)	**41.** (a), (d)	**42.** (a), (c)
43. (a), (c)	**44.** (a), (d)	**45.** (a), (b), (c), (d)	**46.** (b)	**47.** (b)	
48. (b)	**49.** (b)	**50.** (c)	**51.** (c)	**52.** (a), (c), (d)	**53.** (c)
54. (a), (c)	**55.** (b)				

Practice Test

1. (a)	**2.** (c), (d)	**3.** (d)	**4.** (a), (b), (d)	**5.** (c)	**6.** (c)
7. (a), (c)	**8.** (b)	**9.** (b), (c)	**10.** (d)		

13

DIFFERENTIAL COEFFICIENT

§ 13.1. Theorems on Derivatives

(i) $\frac{d}{dx}\{f_1(x) \pm f_2(x)\} = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)$

(ii) $\frac{d}{dx}(k\,f(x)) = k\frac{d}{dx} f(x)$, where k is any constant

(iii) $\frac{d}{dx}\{f_1(x) \cdot f_2(x)\} = f_1(x)\frac{d}{dx} f_2(x) + f_2(x)\frac{d}{dx} f_1(x)$.

In particular Chain Rule

$$\frac{d}{dx}\{f_1(x) \cdot f_2(x) \cdot f_3(x) \ldots\ldots\} = \left(\frac{d}{dx} f_1(x)\right)(f_2(x)\, f_3(x) \ldots\ldots) + \left(\frac{d}{dx} f_2(x)\right)(f_1(x)\, f_3(x) \ldots\ldots) + \left(\frac{d}{dx} f_3(x)\right)(f_1(x)).\, f_2(x) \ldots\ldots\ldots) + \ldots\ldots$$

(iv) $\frac{d}{dx}\left\{\frac{f_1(x)}{f_2(x)}\right\} = \frac{f_2(x) \cdot \frac{d}{dx} f_1(x) - f_1(x) \cdot \frac{d}{dx} f_2(x)}{[f_2(x)]^2}$

(v) If $y = f_1(u)$, $u = f_2(v)$ and $v = f_3(x)$

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

§ 13.2. Derivative of Parametric Equations

If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{g'(t)}{f'(t)}$.

§ 13.3. Derivative of Implicit Functions

If $f(x, y) = 0$, then on differentiating of $f(x, y)$ w.r.t. x, we get $d/dx\, f(x, y) = 0$, Collect the terms of dy/dx and solve.

Alternative Method :

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

In particular

if $f(x_1, x_2, x_3, \ldots\ldots, x_n) = 0$ and $x_2, x_3, \ldots\ldots, x_n$ are the functions of x_1 then

$$\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dx_1} + \frac{\partial f}{\partial x_3} \cdot \frac{dx_3}{dx_1} + \ldots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dx_1}$$

§ 13.4. Derivative of Logarithmic Functions

If $$y = [f_1(x)]^{f_2(x)} \text{ or } y = f_1(x) \cdot f_2(x) \cdot f_3(x) \ldots\ldots$$

or $$y = \frac{f_1(x) \cdot f_2(x) \cdot f_3(x) \ldots\ldots}{g_1(x) \cdot g_2(x) \cdot g_3(x) \ldots\ldots}$$

then it is convenient to take the logarithm of the function first and then differentiate. This is called derivative of logarithmic function

Note : Write $[f(x)]^{g(x)} = e^{g(x) \ln(f(x))}$ and differentiate easily

§ 13.5. Some Standard Substitutions

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$ or $a \cot\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a \cos\theta$ or $a \cos 2\theta$
$\sqrt{(2ax - x^2)}$	$x = a(1 - \cos\theta)$

§ 13.6. Critical Points

The points on the curve $y = f(x)$ at which $dy/dx = 0$ or dy/dx does not exist are known as the critical points.

§ 13.7. Rolle's Theorem

If a function $f(x)$ is defined on $[a, b]$ satisfying (i) f is continuous on $[a, b]$ (ii) f is differentiable on (a, b) (iii) $f(a) = f(b)$ then $c \in (a, b)$ such that $f'(c) = 0$.

§ 13.8. Lagrange's mean value Theorem

If a function $f(x)$ is defined on $[a, b]$ satisfying

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b) then $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

§ 13.9. Test for the Constancy of a Function

If at all points of a certain interval $f'(x) = 0$, then the function $f(x)$ preserves a constant value within this interval.

§ 10.10. $\phi - \Psi$ Method

$$\frac{d}{dx}\int_{\phi(x)}^{\Psi(x)} f(t)\,dt = f\{\Psi(x)\} \cdot \frac{d\Psi}{dx} - f\{\phi(x)\} \cdot \frac{d\phi}{dx}$$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is equal to

(a) $(1+\log x)^{-1}$
(b) $(1+\log x)^{-2}$
(c) $\log x\,(1+\log x)^{-2}$
(d) None of these

2. If $\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2} = k$, then k is equal to

(a) 0 (b) 1
(c) 2 (d) none of these

3. If $y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$ then $\frac{dy}{dx}$ at $x=0$ is

(a) 0 (b) –1
(c) 1 (d) None of these

4. If $x = e^{y+e^{y+e^{y+\ldots\infty}}}$ then $\frac{dy}{dx}$ is

(a) $\frac{1}{x}$ (b) $\frac{1-x}{x}$
(c) $\frac{x}{1+x}$ (d) None of these

5. The derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ is

(a) 0 (b) 1
(c) $\frac{1}{1-x^2}$ (d) $\frac{1}{1+x^2}$

6. If $x^y \cdot y^x = 16$ then $\frac{dy}{dx}$ at (2, 2) is

(a) –1 (b) 0
(c) 1 (d) None of these

7. If $y^2 = P(x)$ is a polynomial of degree 3 then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ equals

(a) $P'''(x) + P'x$ (b) $P''(x)\,.\,P'''(x)$
(c) $P(x).\,P'''(x)$ (d) None of these

8. If $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ and $y = x f(x)$ then $\left(\frac{dy}{dx}\right)_{x=1}$ is equal to

(a) 14 (b) 7/8
(c) 1 (d) None of these

9. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = y = 1$ is

(a) 0 (b) –1
(c) 1 (d) 2

10. If $f(x) = |x-2|$ and $g(x) = fof(x)$, then for $x > 20$, $g'(x) =$

(a) 2 (b) 1
(c) 3 (d) None of these

11. If $f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\sin x)$ and $\phi(x) = f(f(f(x)))$, then $\phi'(x)$

(a) 1 (b) $\sin x$
(c) 0 (d) None of these

12. If $f(x) = (\log_{\cot x}\tan x)(\log_{\tan x}\cot x)^{-1} + \tan^{-1}\left(\frac{x}{\sqrt{4-x^2}}\right)$ then $f'(0)$ is equal to

(a) –2 (b) 2
(c) 1/2 (d) 0

13. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $x^3y\frac{dy}{dx}$ equals

(a) –1 (b) 0
(c) 1 (d) None of these

14. If variables x and y are related by the equation $x = \int_0^y \frac{1}{\sqrt{(1+9u^2)}}\,du$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{\sqrt{1+9y^2}}$ (b) $\sqrt{1+9y^2}$
(c) $1+9y^2$ (d) $\frac{1}{1+9y^2}$

15. If $y^{1/n} = [x + \sqrt{1 + x^2}]$, then $(1 + x^2) y_2 + xy_1$ is equal to

(a) $n^2 y$ (b) ny^2
(c) $n^2 y^2$ (d) None of these

16. If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$ then $f'(1)$ is

(a) –1 (b) 1
(c) log 2 (b) –log 2

17. The solution set of $f'(x) > g'(x)$ where $f(x) = (1/2)\, 5^{2x+1}$ and $g(x) = 5^x + 4x \log 5$ is

(a) $(1, \infty)$ (b) $(0, 1)$
(c) $[0, \infty)$ (d) $(0, \infty)$

18. If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$ then $f^2(19) + g^2(19)$ is

(a) 16 (b) 32
(c) 64 (d) None of these

19. If $\phi(x)$ be a polynomial function of the second degree. If $\phi(1) = \phi(-1)$ and a_1, a_2, a_3 are in A.P. then $\phi'(a_1), \phi'(a_2), \phi'(a_3)$ are in

(a) AP (b) GP
(c) HP (d) None of these

20. If $y = \sin x^\circ$ and $u = \cos x$ then $\frac{dy}{du}$ is equal to

(a) $-\operatorname{cosec} x . \cos x$
(b) $\frac{\pi}{180} \operatorname{cosec} x^\circ . \cos x$
(c) $-\frac{\pi}{180} \operatorname{cosec} x . \cos x^\circ$
(d) None of these

21. The diff. coeffi. of $f(\log x)$ w.r.t x, where $f(x) = \log x$ is

(a) $x/\log x$ (b) $\log x/x$
(c) $(x \log x)^{-1}$ (d) None of these

22. If $y = \sqrt{\left(\frac{1 + \cos 2\theta}{1 - \cos 2\theta}\right)}$, $\frac{dy}{dx}$ at $\theta = 3\pi/4$ is

(a) –2 (b) 2
(c) ± 2 (d) None of these

23. If $y = \sqrt{\sin x + \sqrt{\sin + \sqrt{\sin x + \ldots \infty}}}$ then $\frac{dy}{dx} =$

(a) $\frac{2y-1}{\cos x}$ (b) $\frac{\cos x}{2y-1}$
(c) $\frac{2x-1}{\cos y}$ (d) $\frac{\cos y}{2x-1}$

24. Let $f(x) = \log\left\{\frac{u(x)}{v(x)}\right\}$, $u'(2) = 4$, $v'(2) = 2$, $u(2) = 2$, $v(2) = 1$, then $f'(2)$ is equal to

(a) 0 (b) 1
(c) –1 (d) None of these

25. If $\sqrt{(x^2 + y^2)} = ae^{\tan^{-1}(y/x)}$, $a > 0$ then $y''(0)$ is

(a) $\frac{a}{2} e^{-\pi/2}$ (b) $ae^{\pi/2}$
(c) $-\frac{2}{a} e^{-\pi/2}$ (d) not exist

26. If $f'(x) = \sin x + \sin 4x . \cos x$ then $f'\left(2x^2 + \frac{\pi}{2}\right)$ at $x = \sqrt{\frac{\pi}{2}}$ is equal to

(a) –1 (b) 0
(c) $-2\sqrt{2\pi}$ (d) None of these

27. If $P(x)$ is a polynomial such that $P(x^2 + 1) = \{P(x)\}^2 + 1$ and $P(0) = 0$ then $P'(0)$ is equal to

(a) –1 (b) 0
(c) 1 (d) None of these

28. If $\sin y = x \sin(a + y)$ and $\frac{dy}{dx} = \frac{A}{1 + x^2 - 2x \cos a}$ then the value of A is

(a) 2 (b) $\cos a$
(c) $\sin a$ (d) None of these

29. The third derivative of a function $f(x)$ varishes for all x. If $f(0) = 1, f'(1) = 2$ and $f'' = -1$, then $f(x)$ is equal to

(a) $(-3/2)\, x^2 + 3x + 9$
(b) $(-1/2)\, x^2 - 3x + 1$
(c) $(-1/2)\, x^2 + 3x + 1$
(d) $(-3/2)\, x^2 - 7x + 2$

30. If $y = \log_{e^x} (x - 2)^2$ for $x \neq 0, 2$ then $y'(3)$ is equal to

(a) 1/3 (b) 2/3
(c) 4/3 (d) None of these

MULTIPLE CHOICE–II

Each question, in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

31. Let $f(x) = x^n$, n being a non-negative integer, the value of n for which the equality $f'(a+b) = f'(a) + f'(b)$ is valid for all $a, b > 0$ is

(a) 0 (b) 1
(c) 2 (d) None of these

32. If $y = \tan^{-1}\left(\frac{1}{1+x+x^2}\right) + \tan^{-1}\left(\frac{1}{x^2+3x+3}\right) + \tan^{-1}\left(\frac{1}{x^2+5x+7}\right) + \ldots +$ up to n terms, then $y'(0)$ is equal to

(a) $-\frac{1}{1+n^2}$ (b) $-\frac{n^2}{1+n^2}$
(c) $\frac{n}{1+n^2}$ (d) None of these

33. If $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$ then

(a) $f(0) + f(2) = f(1)$
(b) $f(0) + f(3) = 0$
(c) $f(1) + f(3) = f(2)$
(d) None of these

34. Let f be a function such that $f(x+y) = f(x) + f(y)$ for all x and y and $f(x) = (2x^2 + 3x)\, g(x)$ for all x where $g(x)$ is continuous and $g(0) = 3$. Then $f'(x)$ is equal to

(a) 9 (b) 3
(c) 6 (d) none of these

35. If $\sqrt{(x+y)} + \sqrt{(y-x)} = a$, then $\frac{d^2y}{dx^2}$ equals

(a) $2/a$ (b) $-2/a^2$
(c) $2/a^2$ (d) None of these

36. If $\sin(x+y) = \log(x+y)$, then $\frac{dy}{dx} =$

(a) 2 (b) –2
(c) 1 (d) –1

37. If $y = \log_2 \{\log_2 x\}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\log_2 e}{x \log_e x}$ (b) $\frac{1}{x \log_e x \log_e 2}$
(c) $\frac{1}{\log_e (2x)^x}$ (d) None of these

38. If $y = \frac{\sqrt{(1+t^2)} - \sqrt{(1-t^2)}}{\sqrt{(1+t^2)} + \sqrt{(1-t^2)}}$ and $x = \sqrt{(1-t^4)}$, then $\frac{dy}{dx} =$

(a) $\frac{-1}{t^2\{1+\sqrt{1-t^4}\}}$
(b) $\frac{\{\sqrt{(1-t^4)} - 1\}}{t^6}$
(c) $\frac{1}{t^2\{1+\sqrt{(1-t^4)}\}}$
(d) $\frac{1-\sqrt{(1-t^4)}}{t^6}$

39. Let $x^{\cos y} + y^{\cos x} = 5$. Then

(a) at $x = 0,\ y = 0,\ y' = 0$
(b) at $x = 0,\ y = 1,\ y' = 0$
(c) at $x = y = 1,\ y' = -1$
(d) at $x = 1,\ y = 0,\ y' = 1$

40. If $f(x) = (1+x)^n$ then the value of $f(0) + f'(0) + \frac{f''(0)}{2!} + \ldots + \frac{f^n(0)}{n!}$ is

(a) n (b) 2^n
(c) 2^{n-1} (d) None of these

41. If $f(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$ then the value of $\frac{d^n}{dx^n}(f(x))$ at $x = 0$ for $n = 2m+1$ is

(a) –1 (b) 0
(c) 1 (d) independent of a

42. If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then

(a) f is derivable for all x, with $|x| < 1$
(b) f is not derivable at $x = 1$
(c) f is not derivable at $x = -1$
(d) f is derivable for all x, with $|x| > 1$

43. If $P(x)$ be a polynomial of degree 4, with $P(2) = -1, P'(2) = 0,$
$P''(2) = 2, P'''(2) = -12$ and $P^{iv}(2) = 24$ then $P''(1)$ is equal to

(a) 22 (b) 24
(c) 26 (d) 28

44. Let $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = x^2 + xf'(2) + f''(3)$ then

(a) $f'(1) = 4 + f'(2)$
(b) $g'(2) = 8 + g'(1)$
(c) $g''(2) + f''(3) = 4$
(d) None of these

45. If $f(x) = \sin\left\{\frac{\pi}{3}[x] - x^2\right\}$ for $2 < x < 3$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'(\sqrt{\pi/3})$ is equal to

(a) $\sqrt{\pi/3}$ (b) $-\sqrt{\pi/3}$
(c) $-\sqrt{\pi}$ (d) None of these

Practice Test

M.M. 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s). [10 × 2 = 20]

1. If $y = \int_0^x f(t)\sin\{k(x-t)\}\,dt$, then $\frac{d^2y}{dx^2} + k^2y$ equals

(a) 0 (b) y
(c) $k \cdot f(x)$ (d) $k^2 f(x)$

2. If $\sqrt{(1-x^6)} + \sqrt{(1-y^6)} = a(x^3 - y^3)$ and $\frac{dy}{dx} - f(x, y)\sqrt{\left(\frac{1-y^6}{1-x^6}\right)}$ then

(a) $f(x, y) = y/x$ (b) $f(x, y) = y^2/x^2$
(c) $f(x, y) = 2y^2/x^2$ (d) $f(x, y) = x^2/y^2$

3. If $y = \tan^{-1}\left(\frac{\log_e(e/x^2)}{\log_e(ex^2)}\right) + \tan^{-1}\left(\frac{3 + 2\log_e x}{1 - 6\log_e x}\right)$ then $\frac{d^2y}{dx^2}$ is

(a) 2 (b) 1
(c) 0 (d) –1

4. If $x = a\cos\theta, y = b\sin\theta$, then $\frac{d^3y}{dx^3}$ is equal to

(a) $\left(\frac{-3b}{a^3}\right)\operatorname{cosec}^4\theta\cot^4\theta$
(b) $\left(\frac{3b}{a^3}\right)\operatorname{cosec}^4\theta\cot\theta$
(c) $\left(\frac{-3b}{a^3}\right)\operatorname{cosec}^4\theta\cot\theta$
(d) None of these

5. If $F(x) = \frac{1}{x^2}\int_4^x \{4t^2 - 2F'(t)\}\,dt$, then $F'(4)$ equals

(a) 32/9 (b) 64/3
(c) 64/9 (d) None of these

6. If $f(x-y), f(x).f(y)$ and $f(x+y)$ are in A.P., for all x, y and $f(0) \neq 0$, then

(a) $f(2) = f(-2)$
(b) $f(3) + f(-3) = 0$
(c) $f'(2) + f'(-2) = 0$
(d) $f'(3) = f'(-3)$

7. If $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \ldots\infty}}}}$ then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{2y-1}$ (b) $\frac{y^2 - x}{2y^3 - 2xy - 1}$
(c) $(2y - 1)$ (d) None of these

8. The derivative of $\cos^{-1}\left(\frac{x^{-1} - x}{x^{-1} + x}\right)$ at $x = -1$ is

(a) –2 (b) –1
(c) 0 (d) 1

9. If $\int_{\pi/2}^{x}\sqrt{(3 - 2\sin^2 t)}\,dt + \int_0^y \cos t\,dt = 0$, then $\left(\frac{dy}{dx}\right)_{\pi,\pi}$ is

(a) –3 (b) 0
(c) $\sqrt{3}$ (d) None of these

10. If $f_n(x) = e^{f_{n-1}(x)}$ for all $n \in N$ and $f_0(x) = x$ then $\frac{d}{dx}\{f_n(x)\}$ is equal to

(a) $f_n(x) \cdot \frac{d}{dx}\{f_{n-1}(x)\}$

(b) $f_n(x) . f_{n-1}(x)$

(c) $f_n(x) . f_{n-1}(x) \ldots f_2(x) . f_1(x)$

(d) $\prod_{i=1}^{n} f_i(x)$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a) **2.** (a) **3.** (c) **4.** (b) **5.** (b) **6.** (a)
7. (c) **8.** (b) **9.** (b) **10.** (b) **11.** (c) **12.** (c)
13. (c) **14.** (b) **15.** (a) **16.** (a) **17.** (d) **18.** (b)
19. (a) **20.** (c) **21.** (c) **22.** (b) **23.** (b) **24.** (a)
25. (c) **26.** (c) **27.** (c) **28.** (c) **29.** (c) **30.** (b)

Multiple Choice –II

31. (a), (c) **32.** (b) **33.** (a), (b), (c) **34.** (a) **35.** (c) **36.** (d)
37. (a), (b) **38.** (a), (b) **39.** (c) **40.** (b) **41.** (b), (d)
42. (a), (b), (c), (d) **43.** (c) **44.** (a), (b), (c) **45.** (b)

Practice Test

1. (c) **2.** (d) **3.** (c) **4.** (c) **5.** (a) **6.** (a), (c)
7. (b) **8.** (b) **9.** (c) **10.** (a), (c), (d)

14

TANGENT AND NORMAL

§ 14.1. Tangent and Normal

The equation of the tangent at a point (x_0, y_0) to the curve $y = f(x)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0,\, y_0)} (x - x_0)$$

equation of the normal at a point (x_0, y_0) to the curve $y = f(x)$ is

$$y - y_0 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}} (x - x_0)$$

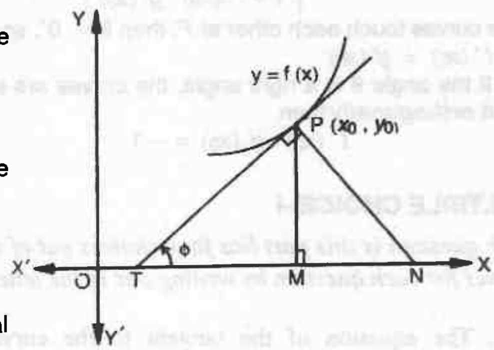


Fig.14.1.

Note : (i) If tangent is parallel to the x-axis or normal is perpendicular to x-axis then $\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)} = 0.$

(ii) If tangent is perpendicular to the x-axis or normal is parallel to the x-axis then $\left(\frac{dx}{dy}\right)_{(x_0,\, y_0)} = 0.$

Length of the Tangent : $$l(PT) = \frac{y_0 \sqrt{1 + \left[\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}\right]^2}}{\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}}$$

Length of the Normal : $$l(PN) = y_0 \sqrt{1 + \left[\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}\right]^2}$$

Length of the Subtangent : $$l(TM) = \frac{y_0}{\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}}$$

Length of the Subnormal : $$l(MN) = y_0 \cdot \left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}$$

§ 14.2. Equations of Tangent and Normal if the Equation of the Curve in Parametric Form

If $$x = f(t), \text{ and } y = g(t)$$

then $$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

In this case, the equations of the tangent and the normal are given by

$$y - g(t) = \frac{g'(t)}{f'(t)} (x - f(t))$$

and $$(y - g(t))\, g'(t) + (x - f(t))\, f'(t) = 0.$$

§ 14.3. The angle between two Curves

Let $y = f(x)$ and $y = g(x)$ be two curves and $P(x_0, y_0)$ be a common point of intersection.

Then the angle between the two curves at $P(x_0, y_0)$ is defined as the angle between the tangents to the curve at P and it is given by

$$\tan\theta = \left|\frac{f'(x_0) - g'(x_0)}{1 + f'(x_0)\cdot g'(x_0)}\right|$$

If the curves touch each other at P, then $\theta = 0°$; so that $f'(x_0) = g'(x_0)$

If the angle θ is a right angle, the curves are said to cut ***orthogonally*** then

$$f'(x_0)\cdot g'(x_0) = -1$$

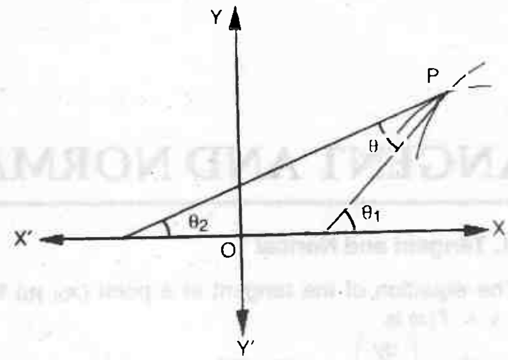


Fig.14.2

MULTIPLE CHOICE–I

Each question is this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. The equation of the tangent to the curve $y = 1 - e^{x/2}$ at the point of intersection with the y-axis is
(a) $x + 2y = 0$ (b) $2x + y = 0$
(c) $x - y = 2$ (d) None of these

2. The line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point
(a) $\left(a, \frac{b}{a}\right)$ (b) $\left(-a, \frac{b}{a}\right)$
(c) $\left(a, \frac{a}{b}\right)$ (d) None of these

3. The chord joining the points where $x = p$ and $x = q$ on the curve $y = ax^2 + bx + c$ is parallel to the tangent at the point on the curve whose abscissa is
(a) $1/2(p + q)$ (b) $1/2\,(p - q)$
(c) $\frac{pq}{2}$ (d) None of these

4. If the tangent to the curve $xy + ax + by = 0$ at $(1, 1)$ is inclined at an angle $\tan^{-1}2$ with x-axis, then
(a) $a = 1, b = 2$ (b) $a = 1, b = -2$
(c) $a = -1, b = 2$ (d) $a = -1, b = -2$

5. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ cut each other orthogonally, then
(a) $a^2 + b^2 = \alpha^2 + \beta^2$
(b) $a^2 - b^2 = \alpha^2 - \beta^2$
(c) $a^2 - b^2 = \alpha^2 + \beta^2$
(d) $a^2 + b^2 = \alpha^2 - \beta^2$

6. The slope of the tangent to the curve $y = \int_0^x \frac{dx}{1 + x^3}$ at the point where $x = 1$ is
(a) 1/4 (b) 1/2
(c) 1 (d) None of these

7. The equation of the tangent to the curve $y = e^{-|x|}$ at the point where the curve cuts the line $x - 1$ is
(a) $x + y = e$ (b) $e(x + y) = 1$
(c) $y + ex = 1$ (d) None of these

8. If the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at any point on it cuts the axes OX and OY at P and Q respectively then $OP + OQ$ is
(a) $a/2$ (b) a
(c) $2a$ (d) $4a$

9. If the tangent at $(1, 1)$ on $y^2 = x(2 - x)^2$ meets the curve again at P, then P is
(a) (4, 4) (b) (–1, 2)
(c) (9/4, 3/8) (d) None of these

10. The tangent to the curve $x = a\sqrt{(\cos 2\theta)}\cos\theta$, $y = a\sqrt{\cos 2\theta}\sin\theta$ at the point corresponding to $\theta = \pi/6$ at
(a) parallel to the x-axis

(b) parallel to the y-axis
(c) parallel to line $y = x$
(d) None of these

11. The area of the triangle formed by the positive x-axis, and the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is
(a) $\sqrt{3}$ (b) $2\sqrt{3}$
(c) $4\sqrt{3}$ (d) None of these

12. The distance between the origin and the normal to the curve $y = e^{2x} + x^2$ at $x = 0$ is
(a) $2/\sqrt{3}$ (b) $2/\sqrt{5}$
(c) $2/\sqrt{7}$ (d) none of these

13. The value of m for which the area of the triangle included between the axes and any tangent to the curve $x^m y = b^m$ is constant, is
(a) 1/2 (b) 1
(c) 3/2 (d) 2

14. The acute angles between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection is
(a) $\pi/4$ (b) $\tan^{-1}(4\sqrt{2}/7)$
(c) $\tan^{-1}(4\sqrt{7})$ (d) none of these

15. If $f(x) = x/\sin x$ and $g(x) = x/\tan x$ where $0 < x \le 1$ then in this interval
(a) both $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

16. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$. Then
(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

17. The normal to the curve represented parametrically by $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ at any point θ, is such that it
(a) makes a constant angle with the x-axis
(b) is at a constant distance from the origin
(c) touches a fixed circle
(d) passes through the origin

18. If the tangent at any point on the curve $x^4 + y^4 = a^4$ cuts off intercepts p and q on the co-ordinate axes, the value of $p^{-4/3} + q^{-4/3}$ is
(a) $a^{-4/3}$ (b) $a^{-1/2}$
(c) $a^{1/2}$ (d) None of these

19. If $y = f(x)$ be the equation of a parabola which is touched by the line $y = x$ at the point where $x = 1$. Then
(a) $f'(0) = f'(1)$
(b) $f'(1) = 1$
(c) $f(0) + f'(0) + f''(0) = 1$
(d) $2f(0) = 1 - f'(0)$

20. Let the parabolas $y = x^2 + ax + b$ and $y = x(c - x)$ touch each other at the point (1, 0) then
(a) $a = -3$ (b) $b = 1$
(c) $c = 2$ (d) $b + c = 3$

21. The point of intersection of the tangents drawn to the curve $x^2 y = 1 - y$ at the points where it is met by the curve $xy = 1 - y$, is given by
(a) (0, –1) (b) (1, 1)
(c) (0, 1) (d) none of these

22. If $y = 4x - 5$ is a tangent to the curve $y^2 = px^3 + q$ at (2, 3) then
(a) $p = 2, q = -7$
(b) $p = -2, q = 7$
(c) $p = -2, q = -7$
(d) $p = 2, q = 7$

23. The slope of the normal at the point with abscissa $x = -2$ of the graph of the function $f(x) = |x^2 - x|$ is
(a) –1/6 (b) –1/3
(c) 1/6 (d) 1/3

24. The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

25. A point on the ellipse $4x^2 + 9y^2 = 36$ where the tangent is equally inclined to the axes is

(a) $\left(\frac{9}{\sqrt{3}}, \frac{4}{\sqrt{13}}\right)$ (b) $\left(-\frac{9}{\sqrt{13}}, \frac{4}{\sqrt{13}}\right)$
(c) $\left(\frac{9}{\sqrt{3}}, -\frac{4}{\sqrt{13}}\right)$ (d) None of these

Practice Test

MM : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s). [10 × 2 = 20]

1. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point
(a) (1, 1) (b) at no point
(c) (0, 1) (d) (1, 0)

2. For $F(x) = \int_0^x 2\,|\,t\,|\,dt$, the tangent lines which are parallel to the bisector of the first co-ordinate angle is
(a) $y = x - \frac{1}{4}$ (b) $y = x + \frac{1}{4}$
(c) $y = x - \frac{3}{2}$ (d) $y = x + \frac{3}{2}$

3. The tangent to the graph of the function $y = f(x)$ at the point with abscissa $x = 1$ form an angle of $\pi/6$ and at the point $x = 2$ an angle of $\pi/3$ and at the point $x = 3$ an angle of $\pi/4$. The value of
$$\int_1^3 f'(x) f''(x)\,dx + \int_2^3 f''(x)\,dx$$
($f''(x)$ is suppose to be continuous) is
(a) $\frac{4\sqrt{3} - 1}{3\sqrt{3}}$ (b) $\frac{3\sqrt{3} - 1}{2}$
(c) $\frac{4 - \sqrt{3}}{3}$ (d) None of these

4. The slope of the tangent to the curve $y = \int_x^{x^2} \cos^{-1} t^2\,dt$ at $x = \frac{1}{\sqrt[4]{2}}$ is
(a) $\left(\frac{\sqrt[4]{8}}{2} - \frac{3}{4}\right)\pi$ (b) $\left(\frac{\sqrt[4]{8}}{3} - \frac{1}{4}\right)\pi$
(c) $\left(\frac{\sqrt[5]{8}}{4} - \frac{1}{3}\right)\pi$ (d) None of these

5. The equations of the tangents to the curve $y = x^4$ from the point (2, 0) not on the curve, are given by
(a) $y = 0$ (b) $y - 1 = 5(x - 1)$
(c) $y - \frac{4098}{81} = \frac{2048}{27}\left(x - \frac{8}{3}\right)$
(d) $y - \frac{32}{243} = \frac{80}{81}\left(x - \frac{2}{3}\right)$

6. If the parametric equation of a curve given by $x = e^t \cot t$, $y = e^t \sin t$, then the tangent to the curve at the point $t = \pi/4$ makes with axis of x the angle
(a) 0 (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

7. For the curve $x = t^2 - 1, y = t^2 - t$, the tangent line is perpendicular to x-axis when
(a) $t = 0$ (b) $t = \infty$
(c) $t = \frac{1}{\sqrt{3}}$ (d) $t = -\frac{1}{\sqrt{3}}$

8. If the subnormal at any point on $y = a^{1-n} x^n$ is of constant length, then the value of n is
(a) –2 (b) 1/2
(c) 1 (d) 2

9. The tangent and normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meet the x-axis in T and G respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through T, P, G is
(a) $\tan^{-1} t^2$ (b) $\cot^{-1} t^2$
(c) $\tan^{-1} t$ (d) $\cot^{-1} t$

10. The value of parameter a so that the line $(3 - a)x + ay + (x^2 - 1) = 0$ is normal to the curve $xy = 1$, may lie in the interval
(a) $(-\infty, 0) \cup (3, \infty)$ (b) (1, 3)
(c) (–3, 3) (d) None of these

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a)	**2.** (d)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (b)
7. (d)	**8.** (b)	**9.** (c)	**10.** (a)	**11.** (b)	**12.** (b)
13. (b)	**14.** (b)	**15.** (c)			

Multiple Choice –II

16. (b), (c)	**17.** (b), (c)	**18.** (a)	**19.** (a), (b), (c) (d)	**20.** (b), (d)
21. (c)	**22.** (d)	**23.** (a)	**24.** (b)	**25.** (a), (b), (c)

Practice Test

1. (d)	**2.** (a), (b)	**3.** (d)	**4.** (b)	**5.** (a), (c)	**6.** (d)
7. (a)	**8.** (b)	**9.** (c)	**10.** (a)		

15

MONOTONOCITY

§ 15.1. Monotonocity

A function f defined on an interval $[a, b]$ said to be

(i) **Monotonically increasing function :**

If $x_2 > x_1 \Rightarrow f(x_2) > f(x_1),\ \forall\ x_1, x_2 \in [a, b]$

(ii) **Strictly increasing function :**

If $x_2 > x_1 \Rightarrow f(x_2) > f(x_1),\ \forall\ x_1, x_2 \in [a, b]$

(iii) **Monotonically decreasing function :**

If $x_2 \geq x_1 \Rightarrow f(x_2) < f(x_1),\ \forall\ x_1, x_2 \in [a, b]$

(iv) **Strictly decreasing function :**

If $x_2 > x_1 \Rightarrow f(x_2) < f(x_1),\ \forall\ x_1, x_2 \in [a, b]$

§ 15.2. Test for Monotonocity

(i) The function $f(x)$ is ***monotonically increasing*** in the interval $[a, b]$, if $f'(x) > 0$ in $[a, b]$.

(ii) The function $f(x)$ is ***strictly increasing*** in the interval $[a, b]$, if $f'(x) > 0$ in $[a, b]$.

(iii) The function $f(x)$ is ***monotonically decreasing*** in the interval $[a, b]$, if $f'(x) < 0$ in $[a, b]$

(iv) The function $f(x)$ is ***strictly decreasing*** in the interval $[a, b]$, if $f'(x) < 0$ in $[a, b]$.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whicheven is appropriate.

1. Let $y = x^2 e^{-x}$, then the interval in which y increases with respect to x is

(a) $(-\infty, \infty)$ (b) $(-2, 0)$

(c) $(2, \infty)$ (d) $(0, 2)$

2. The function $f(x) = \cos\left(\frac{\pi}{x}\right)$ is decreasing in the interval

(a) $(2n + 1, 2n),\ n \in N$

(b) $\left(\frac{1}{2n+1}, 2n\right), n \in N$

(c) $\left(\frac{1}{2n+2}, \frac{1}{2n+1}\right), n \in N$

(d) None of these

3. The set of all values of a for which the function

$$f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x + \log 5$$

decreases for all real x is

(a) $(-\infty, \infty)$

(b) $\left[-4, 3 - \frac{\sqrt{21}}{2}\right] \cup [1, \infty)$

(c) $\left(-3, 5 - \frac{\sqrt{27}}{2}\right) \cup (2, \infty)$

(d) $[1, \infty)$

4. On which of the following intervals is the function $x^{100} + \sin x - 1$ decreasing ?

(a) $(0, \pi/2)$ (b) $(0, 1)$

(c) $\left(\frac{\pi}{2}, \pi\right)$ (d) None of these

5. The function $f(x) = \log(1 + x) - \frac{2x}{2+x}$ is increasing on

(a) $(0, \infty)$ (b) $(-\infty, 0)$

(c) $(-\infty, \infty)$ (d) None of these

6. If $f(x) = \dfrac{a \sin x + b \cos x}{c \sin x + d \cos x}$ is decreasing for all x then
(a) $ad - bc > 0$ (b) $ad - bc < 0$
(c) $ab - cd > 0$ (d) $ab - cd < 0$

7. If $f(x) = (ab - b^2 - 2)\, x + \int_0^x (\cos^4 \theta + \sin^4\theta)\, d\theta$ is decreasing function of x for all $x \in R$ and $b \in R$, b being independent of x, then
(a) $a \in (0, \sqrt{6})$ (b) $a \in (-\sqrt{6}, \sqrt{6})$
(c) $a \in (-\sqrt{6}, 0)$ (d) none of these

8. The value of a in order that $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ decreases for all real values of x, is given by
(a) $a < 1$ (b) $a > 1$
(c) $a \geq \sqrt{2}$ (d) $a < \sqrt{2}$

9. The function $f(x) = \dfrac{ln\,(\pi + x)}{ln\,(e + x)}$ is
(a) increasing on $[0, \infty)$
(b) decreasing on $[0, \infty)$
(c) increasing on $[0, \pi/e)$ and decreasing on $[\pi/e, \infty)$
(d) decreasing on $[0, \pi/e)$ and increasing on $[\pi/e, \infty)$

10. The function f defined by $f(x) = (x + 2)\, e^{-x}$ is
(a) decreasing for all x
(b) decreasing on $(-\infty, -1)$ and increasing in $(-1, \infty)$
(c) increasing for all x
(d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$.

11. If $a < 0$, the function $f(x) = e^{ax} + e^{-ax}$ is a monotonically decreasing function for values of x given by
(a) $x > 0$ (b) $x < 0$
(c) $x > 1$ (d) $x < 1$

12. $y = \{x\,(x - 3)\}^2$ increases for all values of x lying in the interval
(a) $0 < x < 3/2$ (b) $0 < x < \infty$
(c) $-\infty < x < 0$ (d) $1 < x < 3$

13. The function $f(x) = \tan x - x$
(a) always increases
(b) always decreases
(c) never decreases
(d) some times increases and some times decreases

14. If the function $f(x) = \cos |x| - 2ax + b$ increases along the entire number scale, the range of values of a is given by
(a) $a \leq b$ (b) $a = b/2$
(c) $a \leq -1/2$ (d) $a \geq -3/2$

15. The function $f(x) = x\sqrt{(ax - x^2)}$, $a > 0$
(a) increases on the interval $(0, 3a/4)$
(b) decreases on the interval $(3a/4, a)$
(c) decreases on the interval $(0, 3a/4)$
(d) increases on the interval $(3a/4, a$

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

16. Let $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$ for all real values of x. Then
(a) h is increasing whenever $f(x)$ is increasing
(b) h is increasing whenever $f'(x) < 0$
(c) h is decreasing whenever f is decreasing
(d) nothing can be said in general

17. If $\phi(x) = 3f\left(\dfrac{x^2}{3}\right) + f(3 - x^2)\ \ \forall\, x \in (-3, 4)$ where $f''(x) > 0\ \forall\, x \in (-3, 4)$, then $\phi(x)$ is
(a) increasing in $\left(\dfrac{3}{2}, 4\right)$
(b) decreasing in $\left(-3, -\dfrac{3}{2}\right)$
(c) increasing in $\left(-\dfrac{3}{2}, 0\right)$
(d) decreasing in $\left(0, \dfrac{3}{2}\right)$

18. Which of the following functions are decreasing on $(0, \pi/2)$?
(a) $\cos x$ (b) $\cos 2x$
(c) $\cos 3x$ (d) $\tan x$

19. If $\phi(x) = f(x) + f(2a - x)$ and $f''(x) > 0$, $a > 0$, $0 \leq x \leq 2a$ then
(a) $\phi(x)$ increases in $(a, 2a)$
(b) $\phi(x)$ increases in $(0, a)$
(c) $\phi(x)$ decreases in $(a, 2a)$

(d) $\phi(x)$ decreases in $(0, a)$

20. If $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$, then $f(x)$
(a) increases in $(0, \infty)$
(b) decreases in $[0, \infty)$
(c) neither increases nor decreases in $(0, \infty)$
(d) increases in $(-\infty, \infty)$

21. If $f(x) - ax^3 - 9x^2 + 9x + 3$ is increasing on R, then
(a) $a < 3$ (b) $a > 3$
(c) $a \leq 3$ (d) None of these

22. Function $f(x) = |x| - |x - 1|$ is monotonically increasing when
(a) $x < 0$ (b) $x > 1$
(c) $x < 1$ (d) $0 < x < 1$

23. Every invertible function is
(a) monotonic function
(b) constant function
(c) indentity function
(d) not necessarily monotonic function

24. Let $f(x) = 2x^2 - \log|x|, x \neq 0$. Then $f(x)$ is
(a) m.i. in $\left[-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$
(b) m.d. in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$
(c) m.i. in $\left(-\infty, \frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$
(d) m.d. in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

25. Let $f'(x) > 0$ and $g'(x) < 0$ for all $x \in R$. Then
(a) $f\{g(x)\} > f\{g(x+1)\}$
(b) $f\{g(x)\} > f\{g(x-1)\}$
(c) $g\{f(x)\} > g\{f(x+1)\}$
(d) $g\{f(x)\} > g\{f(x-1)\}$

Practice Test

M.M. 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).
$[10 \times 2 = 20]$

1. Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function in the set of real numbers R. Then a and b satisfy the condition
(a) $a^2 - 3b - 15 > 0$
(b) $a^2 - 3b + 15 > 0$
(c) $a^2 - 3b - 15 < 0$
(d) $a > 0$ and $b > 0$

2. Let $Q(x) = f(x) + f(1-x)$ and $f''(x) < 0, 0 < x < 1$, then
(a) Q increases in $[1/2, 1]$
(b) Q decreases in $[1/2, 1]$
(c) Q decreases in $[0, 1/2]$
(d) Q increases in $[0, 1/2]$

3. If $f: R \to R$ is the function defined by
$$f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}, \text{ then}$$
(a) $f(x)$ is an increasing function
(b) $f(x)$ is a decreasing function
(c) $f(x)$ is onto (surjective)
(d) None of these

4. If $f'(x) = |x| - \{x\}$ where $\{x\}$ denotes the fractional part of x, then $f(x)$ is decreasing in
(a) $\left(-\frac{1}{2}, 0\right)$ (b) $\left(-\frac{1}{2}, 2\right)$
(c) $\left(-\frac{1}{2}, 2\right]$ (d) $\left(\frac{1}{2}, \infty\right)$

5. The interval to which b may belong so that the function
$$f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b+1}\right)x^3 + 5x + \sqrt{6}$$
is increasing at every point of its domain is
(a) $[-7, -1]$ (b) $[-6, -2]$
(c) $[2, 2.5]$ (d) $[2, 3]$

6. For $x > 1$, $y = \log_e x$ satisfies the inequality
(a) $x - 1 > y$ (b) $x^2 - 1 > y$
(c) $y > x - 1$ (d) $\frac{x-1}{x} < y$

7. The function $f(x) = x^x$ decreases on the interval
(a) $(0, e)$ (b) $(0, 1)$
(c) $(0, 1/e)$ (d) None of these

8. Let the function $f(x) = \sin x + \cos x$, be defined in $[0, 2\pi]$, then $f(x)$
(a) increases in $(\pi/4, \pi/2)$
(b) decreases in $(\pi/4, 5\pi/4)$
(c) increases in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$
(d) decreases in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, 2\pi\right]$

9. Let $f(x) = \int e^x (x-1)(x-2)\, dx$. Then f decreases in the interval
(a) $(-\infty, -2)$ (b) $(-2, -1)$
(c) $(1, 2)$ (d) $(2, \infty)$

10. For all $x \in (0, 1)$
(a) $e^x > 1 + x$ (b) $\log_e (1+x) < x$
(c) $\sin x > x$ (d) $\log_e x > x$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (d)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (b)
7. (b)	**8.** (b)	**9.** (b)	**10.** (d)	**11.** (a)	**12.** (a)
13. (a)	**14.** (c)	**15.** (a)			

Multiple Choice –II

16. (a), (c)	**17.** (a), (b), (c), (d)		**18.** (a), (b)	**19.** (a), (d)	**20.** (a), (d)
21. (b)	**22.** (d)	**23.** (a)	**24.** (a), (d)	**25.** (a, c)	

Practice Test

1. (c)	**2.** (b), (d)	**3.** (d)	**4.** (a)	**5.** (a), (b), (c), (d)
6. (a), (b), (d)	**7.** (c)	**8.** (b), (c)	**9.** (c)	**10.** (b)

16

MAXIMA AND MINIMA

§ 16.1. Maxima and Minima

From the figure for the function $y = f(x)$, we find that the function gets local maximum and local minimum. The tangent to the curve at these points are parallel to x-axis, *i.e.*, $\frac{dy}{dx} = 0$.

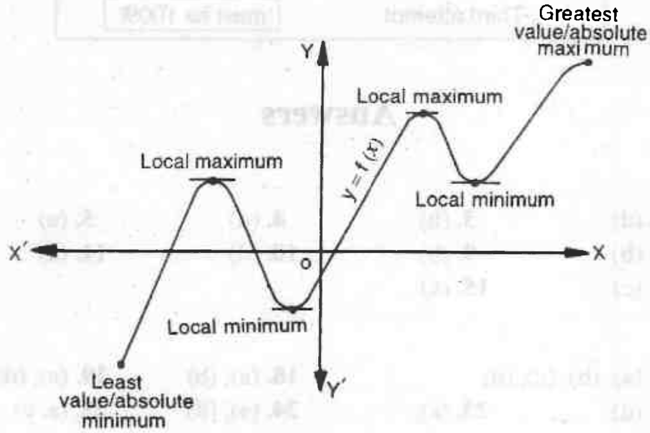


Fig. 16.1.

§ 16.2. Working Rule for Finding Maxima and Minima

(a) First Derivative Test :

To check the maxima or minima at $x = a$

(i) If $f'(x) > 0$ at $x < a$ and $f'(x) < 0$ at $x > a$ *i.e.* the sign of $f'(x)$ changes from + ve to – ve, then $f(x)$ has a local maximum at $x = a$.

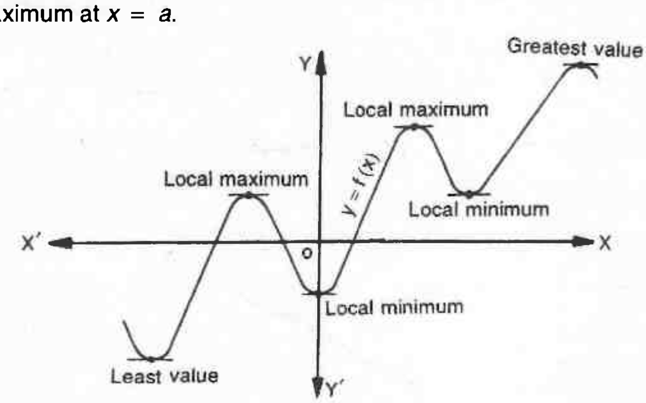


Fig. 16.2.

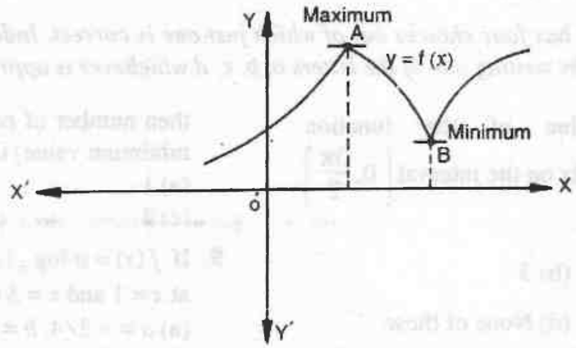


Fig. 16.3.

(ii) If $f'(x) < 0$ at $x < a$ and $f'(x) > 0$ at $x > a$ *i.e.* the sign of $f'(x)$ changes from – ve to + ve, then $f(x)$ has a local minimum at $x = a$.

(iii) If the sign of $f'(x)$ does not change, then $f(x)$ has neither local maximum nor local minimum at $x = a$, then point 'a' is called a point of ***inflexion.***

(b) Second derivative Test :

(i) If $f''(a) < 0$ and $f'(a) = 0$, then 'a' is a point of local maximum.

(ii) If $f''(a) > 0$ and $f'(a) = 0$, then 'a' is a point of local minimum.

(iii) If $f''(a) = 0$ and $f'(a) = 0$ then further differentiate and obtain $f'''(a)$.

(iv) If $f'(a) = f''(a) = f'''(a) = ... = f^{n-1}(a) = 0$ and $f^n(a) \neq 0$.

If n is odd then $f(x)$ has neither local maximum nor local minimum at $x = a$, then point 'a' is called a point of ***inflexion.***

If n is even, then if $f^n(a) < 0$ then $f(x)$ has a local maximum at $x = a$ and if $f^n(a) > 0$ then $f(x)$ has a local minimum at $x = a$.

Note : Maximum or minimum values are also called local extremum values. For the points of local extremum either $f'(x) = 0$ or $f'(x)$ does not exist.

§ 16.3. Greatest and Least Values of a Function

Given a function $f(x)$ in an interval $[a, b]$, the value $f(c)$ is said to be

(i) The greatest value of $f(x)$ in $[a, b]$ if $f(c) > f(x)$ for all x in $[a, b]$.

(ii) The least value of $f(x)$ in $[a, b]$ if $f(c) < f(x)$ for all x in $[a, b]$.

Two important tips : (i) If $y = \frac{g(x)}{f(x)}$,

Let $z = \frac{1}{y} = \frac{f(x)}{g(x)}$ {If degree of $g(x) <$ degree of $f(x)$}

if z is minimum $\Rightarrow y$ is maximum.

if z is maximum $\Rightarrow y$ is minimum.

(ii) Stationary value : A function is said to be stationary for $x = c$, then $f(c)$ is called stationary value if $f'(c) = 0$.

MULTIPLE CHOICE – I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. The greatest value of the function $f(x) = 2\sin x + \sin 2x$ on the interval $\left[0, \frac{3\pi}{2}\right]$ is
(a) $\frac{3\sqrt{3}}{2}$ (b) 3
(c) 3/2 (d) None of these

2. The points of extremum of the function $F(x) = \int_1^x e^{-t^2/2}(1-t^2)dt$ are
(a) ± 1 (b) 0
(c) $\pm 1/2$ (d) ± 2

3. A function f such that $f'(2) = f''(2) = 0$ and f has a local maximum of -17 at 2 is
(a) $(x-2)^4$ (b) $3-(x-2)^4$
(c) $-17-(x-2)^4$ (d) None of these

4. The difference between the greatest and the least value of the function
$$f(x) = \int_0^x (t^2+t+1)\,dt \text{ on } [2, 3] \text{ is}$$
(a) 37/6 (b) 47/6
(c) 57/6 (d) 59/6

5. Let $f(x) = a-(x-3)^{8/9}$, then maxima of $f(x)$ is
(a) 3 (b) $a-3$
(c) a (d) None of these

6. Let $f(x)$ be a differential function for all x, If $f(1) = -2$ and $f'(x) > 2$ for all x in $[1, 6]$, then minimum value of $f(6) =$
(a) 2 (b) 4
(c) 6 (d) 8

7. The point in the interval $[0, 2\pi]$ where $f(x) = e^x \sin x$ has maximum slope is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) None of these

8. Let
$$f(x) = \begin{cases} |x^3+x^2+3x+\sin x|\left(3+\sin\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
then number of points (where $f(x)$ attains its minimum value) is
(a) 1 (b) 2
(c) 3 (d) infinite many

9. If $f(x) = a\log_e|x| + bx^2 + x$ has extremum at $x = 1$ and $x = 3$ then
(a) $a = -3/4, b = -1/8$
(b) $a = 3/4, b = -1/8$
(c) $a = -3/4, b = 1/8$
(d) None of these

10. Let $f(x) = 1 + 2x^2 + 2^2x^4 + \ldots + 2^{10}x^{20}$. Then $f(x)$ has
(a) more than one minimum
(b) exactly one minimum
(c) at least one maximum
(d) None of these

11. Let $f(x) = \begin{cases} \sin^{-1}\alpha + x^2, & 0 < x < 1 \\ 2x, & x > 1 \end{cases}$
$f(x)$ can have a minimum at $x = 1$ is the value of α is
(a) 1 (b) -1
(c) 0 (d) None of these

12. A differentiable function $f(x)$ has a relative minimum at $x = 0$ then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for
(a) all a and all b (b) all b if $a = 0$
(c) all $b > 0$ (d) all $a > 0$

13. The function $f(x) = \int_1^x \{2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\}\,dt$ attains its maximum at $x =$
(a) 1 (b) 2
(c) 3 (d) 4

14. Assuming that the petrol burnt in a motor boat varies as the cube of its velocity, the most economical speed when going against a current of c km/hr is
(a) $(3c/2)$ km/hr (b) $(3c/4)$ km/hr
(c) $(5c/2)$ km/hr (d) $(c/2)$ km/hr

15. N Characters of information are held on magnetic tape, in batches of x characters each, the batch processing time is $\alpha + \beta x^2$ seconds, α and β are constants. The optical value of x for fast processing is,

(a) $\dfrac{\alpha}{\beta}$ (b) $\dfrac{\beta}{\alpha}$

(c) $\sqrt{\dfrac{\alpha}{\beta}}$ (d) $\sqrt{\dfrac{\beta}{\alpha}}$

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

16. Let $f(x) = \cos x \sin 2x$ then

(a) $\min\limits_{x \in (-\pi, \pi)} f(x) > -7/9$

(b) $\min\limits_{x \in (-\pi, \pi)} f(x) > -9/7$

(c) $\min\limits_{x \in [-\pi, \pi]} f(x) > -1/9$

(d) $\min\limits_{x \in [-\pi, \pi]} f(x) > -2/9$

17. The minimum value of the function defined by $f(x) = \text{maximum}\{x, x+1, 2-x\}$ is

(a) 0 (b) 1/2

(c) 1 (d) 3/2

18. Let $f(x) = \begin{cases} x^2 + 3x, & -1 \le x < 0 \\ -\sin x, & 0 < x < \pi/2 \\ -1 - \cos x, & \dfrac{\pi}{2} < x \le \pi \end{cases}$

Then global maxima of $f(x)$ equals and global minima of $f(x)$ are

(a) -1 (b) 0

(c) -3 (d) -2

19. On $[1, e]$, the least and greatest values of $f(x) = x^2 \ln x$ is

(a) e, 1 (b) 1, e

(c) 0, e^2 (d) none of these

20. The minimum value of

$$\left(1 + \frac{1}{\sin^n \alpha}\right)\left(1 + \frac{1}{\cos^n \alpha}\right) \text{ is}$$

(a) 1 (b) 2

(c) $(1 + 2^{n/2})^2$ (d) None of these

21. The fuel charges for running a train are proportional to the square of the speed generated in miles per hour and costs Rs. 48 per hour at 16 miles per hour. The most economical speed if the fixed charges *i.e.*, salaries etc. amount to Rs. 300 per hour.

(a) 10 (b) 20

(c) 30 (d) 40

22. The maximum area of the rectangle that can be inscribed in a circle of radius r is

(a) πr^2 (b) r^2

(c) $\dfrac{\pi r^2}{4}$ (d) $2r^2$

23. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha).f(\beta) < 0$ then the equation $f(x) = 0$ has

(a) three equal real roots

(b) three distinct real roots

(c) one positive root if $f(\alpha) < 0$ and $f(\beta) > 0$

(d) one negative root if $f(\alpha) > 0$ and $f(\beta) < 0$

24. Let $f(x)$ be a function such that $f'(a) \neq 0$. Then at $x = a, f(x)$

(a) can not have a maximum

(b) can not have a minimum

(c) must have neither a maximum nor a minimum

(d) none of these

25. A cylindrical gas container is closed at the top and open at the bottom; if the iron plate of the top is 5/4 times as thick as the plate forming the cylindrical sides. The ratio of the radius to the height of the cylinder using minimum material for the same capacity is

(a) 2/3 (b) 1/2

(c) 4/5 (d) 1/3

Practice Test

M.M : 20 Time : 30 Min

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. Let $f(x) = (x^2 - 1)^n (x^2 + x + 1)$ then $f(x)$ has local extremum at $x = 1$ when

(a) $n = 2$ (b) $n = 3$

(c) $n = 4$ (d) $n = 6$

2. The critical points of the function $f'(x)$ where $f(x) = \dfrac{|x-2|}{x^2}$ is

(a) 0 (b) 2

(c) 4 (d) 6

3. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$

the set of values of b for which $f(x)$ have greatest value at $x = 1$ is given by

(a) $1 < b < 2$ (b) $b = \{1, 2\}$

(c) $b \in (-\infty, -1)$ (d) None of these

4. Let $f(x) = \int_0^x \frac{\cos t}{t} dt \ (x > 0)$; then $f(x)$ has

(a) maxima when $n = -2, -4, -6, \ldots$

(b) maxima when $n = -1, -3, -5, \ldots$

(c) minima when $n = 0, 2, 4, \ldots$

(d) minima when $n = 1, 3, 5, \ldots$

5. Let $f : [a, b] \to R$ be a function such that for $c \in (a, b), f'(c) = f''(c) = f'''(c) = f^{iv}(c) = f^{v}(c) = 0$ then

(a) f has local extremum at $x = c$

(b) f has neither local maximum nor local minimum at $x = c$

(c) f is necessarily a constant function

(d) it is difficult to say whether (a) or (b)

6. From the graph we can conclude that the

(a) Function has some roots

(b) Function has interval of increase and decrease

(c) Greatest and the least values of the function exist

(d) Function is periodic

7. Let $f(x) = (x-1)^m (x-2)^n, x \in R$, Then each critical point of $f(x)$ is either local maximum or local minimum where

(a) $m = 2, n = 3$ (b) $m = 2, n = 4$

(c) $m = 3, n = 4$ (d) $m = 4, n = 2$

8. The number of solutions of the equation $a^{f(x)} + g(x) = 0$, where $a > 0, g(x) \ne 0$ and has minimum value 1/2 is

(a) one (b) two

(c) infinite many (d) zero

9. Let $f(x) = \begin{cases} \sin \frac{\pi x}{2}, & 0 < x < 1 \\ 3 - 2x, & x > 1 \end{cases}$ then

(a) $f(x)$ has local maxima at $x = 1$

(b) $f(x)$ has local minima at $x = 1$

(c) $f(x)$ does not have any local extrema at $x = 1$

(d) $f(x)$ has global minima at $x = 1$

10. Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by all 200 students is to be as small as possible, then the school should be built at

(a) town B (b) 45 km from town A

(c) town A (d) 45 km from town B

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice – I

1. (a)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (b)
7. (b)	**8.** (a)	**9.** (a)	**10.** (b)	**11.** (d)	**12.** (b)
13. (a)	**14.** (a)	**15.** (c)			

Multiple Choice –II

16. (a), b)	**17.** (d)	**18.** (b), (d)	**19.** (c)	**20.** (c)	**21.** (c)
22. (d)	**23.** (b), (c), (d)	**24.** (d)	**25.** (d)		

Practice Test

1. (a), (c), (d)	**2.** (a), (b), (d)	**3.** (d)	**4.** (a), (d)	**5.** (d)	**6.** (b)
7. (b), (d)	**8.** (d)	**9.** (a)	**10.** (c)		

17

INDEFINITE INTEGRATION

17.1. Methods of Integration

(i) Integration by Substitution (or change of independent variable) :

If the independent variable x in $\int f(x)\,dx$ be changed to t, then we substitute $x = \phi(t)$ *i.e.*, $dx = \phi'(t)\,dt$

$$\therefore \qquad \int f(x)\,dx = \int f(\phi(t))\,\phi'(t)\,dt$$

which is either a standard form or is easier to integrate.

(ii) Integration by parts :

If u and v are the differentiable functions of x then

$$\int u.v\,dx = u\int v\,dx - \int\left[\left(\frac{d}{dx}u\right)\left(\int v\,dx\right)\right]dx.$$

17.2. How to choose Ist and IInd functions :

(i) If the two functions are of different types take that function as Ist which comes first in the word **ILATE** where **I** stands for inverse circular function, **L** stands for logarithmic function **A** stands for Algebraic function, **T** stands for trigonometric function and **E** stands for exponential function.

(ii) If both functions are algebraic take that function as Ist whose differential coefficient is simple.

(iii) If both functions are trigonometrical take that function as IInd whose integral is simpler.

Successive integration by parts :

Use the following formula

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots +$$
$$+ \dots + (-1)^{n-1}u^{n-1}v_n + (-1)^n\int u^n v_n\,dx.$$

where u^n stands for nth differential coefficient of u w.r.t. x and v_n stands for nth integral of v w.r.t. x.

Note : $\int \frac{dx}{\sqrt{(2ax - x^2)}} = \text{vers}^{-1}\left(\frac{x}{a}\right) + c$ $\qquad \left\{\begin{array}{l}\therefore \quad 1 - \cos\theta = \text{vers}\,\theta \\ \quad\;\; 1 - \sin\theta = \text{covers}\,\theta\end{array}\right\}$

Cancellation of Integrals :

i.e.
$$\int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + c.$$

17.3. Evaluation of Integrals of Various Type

Type I : Integrals of the type.

(i) $\int \frac{dx}{ax^2 + bx + c}$, (ii) $\int \frac{dx}{\sqrt{(ax^2 + bx + c)}}$, (iii) $\int \sqrt{(ax^2 + bx + c)}\,dx$

Rule : Express $ax^2 + bx + c$ in the form of perfect square and then apply the standard results.

Type II : Integrals of the type

(i) $\int \frac{px + q}{ax^2 + bx + c}\,dx$ (ii) $\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\,dx$ (iii) $\int \frac{(p_0x^n + p_1x^{n-1} + \dots + p_{n-1}x + p_n)}{(ax^2 + bx + c)}\,dx$

Rule for (i) :
$$\int \frac{(px + q)\,dx}{ax^2 + bx + c} = \frac{p}{(2a)}\int \frac{(2ax + b)\,dx}{(ax^2 + bx + c)} + \left(q - \frac{pb}{2a}\right)\int \frac{dx}{ax^2 + bx + c}$$
$$= \frac{p}{2a}\ln|ax^2 + bx + c| + \left(q - \frac{bp}{2a}\right)\int \frac{dx}{ax^2 + bx + c}$$

The other integral of R.H.S. can be evaluated with the help of type I.

Rule for (ii) :

$$\int \frac{(px+q)\,dx}{\sqrt{(ax^2+bx+c)}} = \frac{p}{2a}\int \frac{(2ax+b)}{\sqrt{(ax^2+bx+c)}}\,dx + \left(q-\frac{pb}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= \frac{p}{a}\sqrt{(ax^2+bx+c)} + \left(q-\frac{pb}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

The other integral of R.H.S. can be evaluated with the help of type I.

Rule for (iii) : In this case by actual division reduce the fraction to the form $f(x)+\dfrac{(px+q)}{(ax^2+bx+c)}$ and then integrate.

Type III : Integrals of the form

(i) $\displaystyle\int \frac{dx}{a+b\sin^2 x}$ (ii) $\displaystyle\int \frac{dx}{a+b\cos^2 x}$ (iii) $\displaystyle\int\frac{dx}{a\sin^2 x+b\cos^2 x}$

(iv) $\displaystyle\int \frac{dx}{(a\sin x+b\cos x)^2}$ (v) $\displaystyle\int \frac{\phi(\tan x)\,dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x+d}$

where $\phi(\tan x)$ is a polynomial in $\tan x$.

Rule : We shall always in such cases divide above and below by $\cos^2 x$; then put $\tan x = t$ *i.e.* $\sec^2 x\,dx = dt$ then the question shall reduce to the forms $\displaystyle\int \frac{dt}{(at^2+bt+c)}$ or $\displaystyle\int \frac{\phi(t)\,dt}{(at^2+bt+c)}$.

Type IV : Integrals of the form

(i) $\displaystyle\int \frac{dx}{a+b\sin x}$ (ii) $\displaystyle\int \frac{dx}{a+b\cos x}$ (iii) $\displaystyle\int \frac{dx}{a\sin x+b\cos x+c}$

(iv) $\displaystyle\int \frac{(p\cos x+q\sin x)}{(a\cos x+b\sin x)}\,dx$ (v) $\displaystyle\int \frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+c}\,dx$

Rule for (i), (ii) and (iii) :

write $\cos x = \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}$, $\sin x = \dfrac{2\tan x/2}{1+\tan^2 x/2}$

the numerator shall become $\sec^2 x/2$ and the denominator will be a quadratic in $\tan x/2$. Putting $\tan x/2 = t$ *i.e.* $\sec^2 x/2\,dx = 2dt$ the question shall reduce to the form $\displaystyle\int \frac{dt}{at^2+bt+c}$

Rule for (iv) : Express the numerator as

$$l(D^r) + m(\text{d.c. of } D^r)$$

find l and m by comparing the coefficients of $\sin x$ and $\cos x$ and split the integral into sum of two integrals as

$$l\int dx + m\int \frac{\text{d.c. of } D^r}{D^r}$$

$$= lx + m\ln|D^r| + c$$

Rule for (v) : Express the numerator as

$l(D^r) + m(\text{d.c. of } D^r) + n$, find l, m, and n by comparing the coefficients of $\sin x$, $\cos x$ and constant term and split the integral into sum of three integrals as

$$l\int dx + m\int \frac{\text{d.c. of } D^r}{D^r}\,dx + n\int \frac{dx}{D^r}$$

$= lx + m\ln|D^r| + n\displaystyle\int \frac{dx}{D^r}$ and to evaluate $\displaystyle\int \frac{dx}{D^r}$ proceed by the method to evaluate rule (i), (ii) and (iii).

Type V : Integrals of the form

(i) $\displaystyle\int \frac{(x^2+a^2)\,dx}{(x^4+kx^2+a^4)}$

(ii) $\displaystyle\int \frac{(x^2-a^2)\,dx}{(x^4+kx^2+a^4)}$

where k is a constant, + ve, – ve or zero.

Rule for (i) and (ii) : Divide above and below by x^2 then putting (i) $t = x - \frac{a^2}{x}$ and (ii) $t = x + \frac{a^2}{x}$

i.e., $$dt = \left(1 + \frac{a^2}{x^2}\right)dx \quad \text{and} \quad dt = \left(1 - \frac{a^2}{x^2}\right)dx$$

then the questions shall reduce to the form $$\int \frac{dt}{t^2 + c^2} \text{ or } \int \frac{dt}{t^2 - c^2}$$

Remember :

(i) $\int \frac{x^2\,dx}{x^4 + kx^2 + a^4} = \frac{1}{2}\int \frac{(x^2 + a^2)\,dx}{(x^4 + kx^2 + x^4)} + \frac{1}{2}\int \frac{(x^2 - a^2)\,dx}{(x^4 + kx^2 + a^4)}$

(ii) $\int \frac{dx}{(x^4 + kx^2 + a^4)} = \frac{1}{2a^2}\int \frac{(x^2 + a^2)\,dx}{(x^4 + kx^2 + a^4)} - \frac{1}{2a^2}\int \frac{(x^2 - a^2)\,dx}{(x^4 + kx^2 + a^4)}$

(iii) $\int \frac{dx}{(x^2 + k)^n} = \frac{x}{k(2n-2)(x^2 + k)^{n-1}} + \frac{(2n-3)}{k(2n-2)}\int \frac{dx}{(x^2 + k)^{n-1}}$.

Type (VI) : Integrals of the form

(i) $\int \frac{dx}{(Ax + B)\sqrt{(ax + b)}}$

(ii) $\frac{dx}{(Ax^2 + Bx + C)\sqrt{(ax + b)}}$

(iii) $\int \frac{dx}{(Ax + B)\sqrt{(ax^2 + bx + c)}}$

(iv) $\int \frac{dx}{(Ax^2 + Bx + C)\sqrt{(ax^2 + bx + c)}}$

Rule for (i) and (ii) : Put $ax + b = t^2$

Rule for (iii) : Put $Ax + B = \frac{1}{t}$

Rule for (iv) : Put $\frac{ax^2 + bx + c}{Ax^2 + Bx + C} = t^2$

Note : The integral $\int \frac{dx}{(Ax + B)^r\sqrt{(ax^2 + bx + c)}}$ where r is positive integer, may also be evaluated by substitution $Ax + B = \frac{1}{t}$

Type VII : Integral of the type $\int x^m (a + bx^n)^p\,dx$, where $m, n, p \in Q$

Case (i) : (i) If $p \in I_+$ then expand by the formula of Newton binomial.

(ii) If $p < 0$, then we put $x = t^k$, where k is the common denominator of the fractions m and n.

Case (ii) : If $\frac{m+1}{n}$ = Integer then we put $a + bx^n = t^\alpha$

where α is the denominator of the fraction p.

Case (iii) : If $\frac{m+1}{n} + p$ = Integer then we put $a + bx^n = t^\alpha x^n$ where α is the denominator of the fraction p.

Type VIII : Integrals of the type :

$$\int f(x, (ax + b)^{p_1/q_1}, (ax + b)^{p_2/q_2}, \ldots\ldots)\,dx$$

where f is irrational and $p_1, p_2, \ldots, q_1, q_2, \ldots \in I$ then put $ax + b = t^m$, where m is the L.C.M. of $q_1, q_2 \ldots$

Type IX : Integral of the form

$$\int \sin^m x \cos^n x \, dx$$

Case (i) : If m is odd and n is even then put $\cos x = t$.
Case (ii) : If m is even and n is odd then put $\sin x = t$
Case (iii) : If m and n both are odd then
if $m > n$, put $\sin x = t$
$m < n$, put $\cos x = t$
$m = n$, put $\sin x = t$ or $\cos x = t$.
Case (iv) : If $m + n = -$ve and even.
Then convert the given integral in terms of $\tan x$ and $\sec x$ then put $\tan x = t$.
Case (v) : If m and n are even integer, then convert them in terms of multiple angles by using the formulae

$$\cos^2 x = \frac{1 + \cos 2x}{2}; \sin^2 x = \frac{1 - \cos 2x}{2}; \sin x \cos x = \frac{\sin 2x}{2}$$

and $\quad 2 \cos x \cos y = \cos (x + y) + \cos (x - y)$

17.4. Standard Substitution

Expression	Substitution
$\sqrt{\frac{(x-\alpha)}{(\beta - x)}}$ or $\sqrt{(x-\alpha)(\beta - x)}$	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$
$\sqrt{\left(\frac{x-\alpha}{x-\beta}\right)}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha \sec^2 \theta - \beta \tan^2 \theta$
$\frac{1}{\sqrt{(x-\alpha)(x-\beta)}}$	$x - \alpha = t^2$ or $x - \beta = t^2$

MULTIPLE CHOICE–I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If a particle is moving with velocity $v(t) = \cos \pi t$ along a straight line such that at $t = 0$, $s = 4$ its position function is given by

(a) $\frac{1}{\pi} \cos \pi t + 2$ (b) $-\frac{1}{\pi} \sin \pi t + 4$

(c) $1/\pi \sin \pi t + 4$ (d) None of these

2. Let $f(x)$ be a function such that, $f(0) = f'(0) = 0, f''(x) = \sec^4 x + 4$, then the function is

(a) $\log (\sin x) + \frac{1}{3} \tan^3 x + xb$

(b) $\frac{2}{3} \log (\sec x) + \frac{1}{6} \tan^2 x + 2x^2$

(c) $\log (\cos x) + \frac{1}{6} \cos^2 x + \frac{x^2}{5}$

(d) None of these

3. $\int \frac{(2x^{12} + 5x^9)}{(x^5 + x^3 + 1)^3} dx$ is equal to

(a) $\frac{x^2 + 2x}{(x^5 + x^3 + 1)^2} + c$

(b) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + c$

(c) $ln (x^5 + x^3 + 1 + \sqrt{2x^7 + 5x^4}) + c$

(d) None of these

4. If $\int f(x) \cos x \, dx = \frac{1}{2} f^2(x) + c$ then $f(x)$ is

(a) x (b) $\sin x$

(c) $\cos x$ (d) 1

5. $\int (\sin 2x - \cos 2x) \, dx = \frac{1}{\sqrt{2}} \sin (2x - a) + b$

(a) $a = \frac{5\pi}{4}, b \in R$ (b) $a = \frac{-5\pi}{4}, b \in R$

(c) $a = \frac{\pi}{4}, b \in R$ (d) None of these

6. The primitive function of the function $f(x) = \dfrac{\sqrt{(a^2 - x^2)}}{x^4}$ is

(a) $c + \dfrac{\sqrt{a^2 - x^2}}{3a^2x^3}$ (b) $c - \dfrac{(a^2 - x^2)^{3/2}}{2a^2x^2}$

(c) $c - \dfrac{(a^2 - x^2)^{3/2}}{3a^2x^3}$ (d) None of these

7. The antiderivative of $f(x) = \dfrac{1}{3 + 5\sin x + 3\cos x}$ whose graph passes through the point (0, 0) is

(a) $\dfrac{1}{5}\left(\log\left|1 - \dfrac{5}{3}\tan x/2\right|\right)$

(b) $\dfrac{1}{5}\left(\log\left|1 + \dfrac{5}{3}\tan x/2\right|\right)$

(c) $\dfrac{1}{5}\left(\log\left|1 + \dfrac{5}{3}\cot x/2\right|\right)$

(d) None of these

8. $\int x^x (1 + \log x)\, dx$ is equal to

(a) $x^x \log_e x + c$ (b) $e^{x^x} + c$

(c) $x^x + c$ (d) None of these

9. $\int (x - a)(x - b)(x - c) \dots (x - z)\, dx$ is equal to

(a) constant (b) $5c + 5d + x$

(c) 0 (d) None of these

10. $\int \dfrac{1}{x^2(x^4 + 1)^{3/4}}\, dx$ is equal to

(a) $\left(1 + \dfrac{1}{x^4}\right)^{1/4} + c$ (b) $(x^4 + 1)^{1/4} + c$

(c) $\left(1 - \dfrac{1}{x^4}\right)^{1/4} + c$ (d) $-\left(1 + \dfrac{1}{x^4}\right)^{1/4} + c$

11. Let the equation of a curve passing through the point (0, 1) be given by $y = \int x^2 . e^{x^3} dx$. If the equation of the curve is written in the form $x = f(y)$ then $f(y)$ is

(a) $\sqrt{ln\left(\dfrac{3y - 2}{3}\right)}$ (b) $\sqrt[3]{ln\left(\dfrac{2 - 3y}{3}\right)}$

(c) $\sqrt[3]{ln\left(\dfrac{3y - 2}{3}\right)}$ (d) None of these

12. $\int \dfrac{xe^x}{(1 + x)^2}\, dx$ is equal to

(a) $\dfrac{e^x}{x + 1} + c$ (b) $e^x (x + 1) + c$

(c) $-\dfrac{e^x}{(x + 1)^2} + c$ (d) $\dfrac{e^x}{1 + x^2} + c$

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

13. If $\int \dfrac{\sin x}{\sin (x - \alpha)}\, dx = Ax + B \log \sin (x - \alpha) + C$ then

(a) $A = \sin \alpha$

(b) $B = \cos \alpha$

(c) $A = \cos \alpha$

(d) $B = \sin \alpha$

14. If the derivative of $f(x)$ w.r.t. x is $\dfrac{\frac{1}{2} - \sin^2 x}{f(x)}$, then $f(x)$ is a periodic function with period

(a) $\pi/2$ (b) π

(c) 2π (d) not defined

15. $\int x^{-2/3} (1 + x^{1/2})^{-5/3}\, dx$ is equal to

(a) $3(1 + x^{-1/2})^{-1/3} + c$

(b) $3(1 + x^{-1/2})^{-2/3} + c$

(c) $3(1 + x^{1/2})^{-2/3} + c$

(d) None of these

16. If $\int \dfrac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\, dx = Ax + B \log_e (9e^{2x} - 4) + C$ then

(a) $A = 3/2$ (b) $B = 35/36$

(c) C is indefinite (d) $A + B = -\dfrac{19}{36}$

17. Let $\int \dfrac{x^{1/2}}{\sqrt{1 - x^3}}\, dx = \dfrac{2}{3} gof(x) + c$ then

(a) $f(x) = \sqrt{x}$ (b) $f(x) = x^{3/2}$

(c) $f(x) = x^{2/3}$ (d) $g(x) = \sin^{-1} x$

18. If $\int \text{cosec } 2x\, dx = f(g(x)) + C$, then

(a) range $g(x) = (-\infty, \infty)$
(b) dom $f(x) = (-\infty, \infty) - \{0\}$
(c) $g'(x) = \sec^2 x$
(d) $f'(x) = 1/x$ for all $x \in (0, \infty)$

19. If $\int \frac{xe^x}{\sqrt{(1+e^x)}} dx = f(x)\sqrt{(1+e^x)} - 2\log g(x) + C$, then
(a) $f(x) = x - 1$
(b) $g(x) = \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1}$
(c) $g(x) = \frac{\sqrt{1+e^x} + 1}{\sqrt{1+e^x} - 1}$
(d) $f(x) = 2(x-2)$

20. $\int \frac{\cos 4x - 1}{\cot x - \tan x} dx$ is equal to
(a) $-\frac{1}{2}\cos 4x + c$ (b) $-\frac{1}{4}\cos 4x + c$
(c) $-\frac{1}{2}\sin 2x + c$ (d) None of these

Practice Test

M.M. : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).
[10 × 2 = 20]

1. If $f(x) = \lim_{n \to \infty} \frac{x^n - x^{-n}}{x^n + x^{-n}}$, $x > 1$ then

$$\int \frac{x f(x) \ln (x + \sqrt{(1+x^2)})}{\sqrt{(1+x^2)}} dx \text{ is}$$

(a) $\ln (x + \sqrt{(1+x^2)}) - x + c$
(b) $\frac{1}{2}\{x^2 \ln (x + \sqrt{(1+x^2)}) - x^2\} + c$
(c) $x \ln (x + \sqrt{(1+x^2)}) - \ln (x + \sqrt{(1+x^2)}) + c$
(d) None of these

2. The value of the integral $\int \frac{dx}{x^2 (1+x^n)^{1/n}}$, $n \in N$ is

(a) $\frac{1}{(1-n)}\left(1 + \frac{1}{x^n}\right)^{1-\frac{1}{n}} + c$
(b) $\frac{1}{(1+n)}\left(1 - \frac{1}{x^n}\right)^{1+\frac{1}{n}} + c$
(c) $-\frac{1}{(1-n)}\left(1 - \frac{1}{x^n}\right)^{1-\frac{1}{n}} + c$
(d) $-\frac{1}{(1+n)}\left(1 + \frac{1}{x^n}\right)^{1+\frac{1}{n}} + c$

3. The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is

(a) $\sin x - 6\tan^{-1}(\sin x) + c$
(b) $\sin x - 2(\sin x)^{-1} + c$
(c) $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + c$
(d) $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + c$

4. If

$$\int f(x) \sin x \cos x \, dx = \frac{1}{2(b^2 - a^2)} \log f(x) + c,$$

then $f(x) =$
(a) $\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$
(b) $\frac{1}{a^2 \sin^2 x - b^2 \cos^2 x}$
(c) $\frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$
(d) None of these

5. If l^r means log log log...... x, the log being repeated r times then $\int \{x\, l(x)\, l^2(x)\, l^3(x) \ldots l^r(x)\}^{-1} dx$ is equal to
(a) $l^{r+1}(x) + c$
(b) $\frac{l^{r+1}(x)}{r+1} + c$
(c) $l^r(x) + c$
(d) None of these

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (c)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (c)
7. (b)	**8.** (c)	**9.** (a)	**10.** (d)	**11.** (c)	**12.** (a)

Multiple Choice –II

13. (c), (d)	**14.** (b)	**15.** (b)	**16.** (b), (c), (d)	**17.** (b), (d)	**18.** (a), (b), (c)
19. (b), (d)	**20.** (d)				

Practice Test

1. (d)	**2.** (a)	**3.** (c)	**4.** (a)	**5.** (a)

18

DEFINITE INTEGRATION

§ 18.1 Definite Integrals

Let f be a function of x defined in the interval $[a, b]$, and let F be another function, such that $F'(x) = f(x)$, for all x in the domain of f, then

$$\int_a^b f(x)\,dx = [F(x)]_a^b$$

$$= F(b) - F(a) \qquad \text{\{Newton-Leibnitz formula\}}$$

is called the definite integral of the function $f(x)$ over the interval $[a, b]$ a and b are called the limits of integration, a being the lower limit and b the upper limit.

Note : In definite integrals constant of integration is never present.

1. Properties of Definite Integrals :

Prop. I : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

Prop. II : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Prop. III : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where c, is a point inside or outside the interval $[a, b]$.

Prop. IV : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Prop. V : $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ or 0

According as $f(x)$ is even or odd function of x.

Prop. VI : $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Prop. VII : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Prop. VIII: $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$, where $f(x)$ is periodic with period T and $n \in I$.

Prop. IX : If $f(a+x) = f(x)$, then

$$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx.$$

Prop. X : $\int_{ma}^{na} f(x)\,dx = (n-m)\int_0^a f(x)\,dx$ If $f(x)$ is a periodic function with period a, i.e., $f(a+x) = f(x)$

Prop. XI : If f is continuous on $[a, b]$, then the integral function g defined by

$g(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$ is differentiable in $[a, b]$ and $g'(x) = f(x)$ for all $x \in [a, b]$.

Prop. XII : If $f(x) < \phi(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\, dx < \int_a^b \phi(x)\, dx$

Prop. XIII : $\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx$

Prop. XIV : If m is the least value and M is the greatest value of the function $f(x)$ on the interval $[a, b]$. (estimation of an integral) then

$$m(b-a) < \int_a^b f(x)\, dx < M(b-a)$$

Prop. XV : (i) If the function $f(x)$ increases and has a concave graph in the interval $[a, b]$, then

$$(b-a)\, f(a) < \int_a^b f(x)\, dx < (b-a)\frac{f(a)+f(b)}{2};$$

(ii) If the function $f(x)$ increases and has a convex graph in the interval $[a, b]$ then

$$(b-a)\frac{f(a)+f(b)}{2} < \int_a^b f(x)\, dx < (b-a)\, f(b).$$

Prop. XVI : If $f(x)$ and $g(x)$ are integrable on the interval (a, b), the **Schwarz-Bunyakovsky** inequality takes place :

$$\left| \int_a^b f(x)\, g(x)\, dx \right| \le \sqrt{\left(\int_a^b f^2(x)\, dx \right)\left(\int_a^b g^2(x)\, dx \right)}$$

Prop. XVII : If a function f is integrable and non-negative on $[a, b]$ and there exists a point $c \in [a, b]$ of continuity of f for which $f(c) > 0$,

$$\text{then} \quad \int_a^b f(x)\, dx > 0 \quad (a < b).$$

Prop. XVIII : Leibniz's Rule :

If f is continuous on $[a, b]$ and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$ then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\, dt = f\{v(x)\}\cdot\frac{dv}{dx} - f\{u(x)\}\cdot\frac{du}{dx}$$

Prop. XIX : If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ at which

$f(c) = \frac{1}{(b-a)}\int_a^b f(x)\, dx$ is called the mean value of the function $f(x)$ on the interval $[a, b]$.

Prop. XX : Given an integral

$$\int_a^b f(x)\, dx$$

where the function $f(x)$ is continuous on the interval $[a, b]$. Introduce a new variable t using the formula $x = \phi(t)$

If (i) $\phi(\alpha) = a$, $\phi(\beta) = b$,

(ii) $\phi(t)$ and $\phi'(t)$ are continuous on $[\alpha, \beta]$,

(iii) $f[\phi(t)]$ is defined and is continuous on $[\alpha, \beta]$, then

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f[\phi(t)]\, \phi'(t)\, dt.$$

Prop. XXI : If $f(x) > 0$ on the interval $[a, b]$, then $\int_a^b f(x) > 0$

Prop. XXII : Suppose that $f(x, \alpha)$ and $f_\alpha'(x, \alpha)$ are continuous functions when $c < \alpha < d$ and $a < x < b$, then

$$I'(\alpha) = \int_a^b f'(x, \alpha)\, dx, \quad \text{where } I'(\alpha)$$

is the derivative of $I(\alpha)$ w.r.t. α and $f'(x, \alpha)$ is the derivative of $f(x, \alpha)$ w.r.t. α, keeping x constant.

Prop. XXIII : If f is continuous on $[a, b]$, then $\int_a^x f(t)\,dt$ for $x \in [a, b]$ is differentiable on $[a, b]$.

Prop. XXIV : Improper integrals :

(i) $\int_a^{\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx.$

(ii) $\int_{-\infty}^{a} f(x)\,dx = \lim_{\alpha \to -\infty} \int_{\alpha}^{a} f(x)\,dx$

(iii) $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$

18.2. Definite Integral as the Limit of a Sum

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. Then

$$\int_a^b f(x)\,dx = \lim_{\substack{h \to 0 \\ n \to \infty \\ nh = b-a}} h \sum_{r=0}^{(n-1)} f(a + rh)$$

18.3. Some Important Results to Remember

(i) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

(ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2.$

(iv) In G.P., sum of n terms, $S_n = \frac{a(r^n - 1)}{(r-1)}, r > 1$

$= \frac{a(1 - r^n)}{(1 - r)}, r < 1$

(v) $\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + \overline{n-1}\,\beta)$

$= \frac{\sin n\beta/2}{\sin \beta/2} \cdot \sin\left(\frac{\text{Ist angle + last angle}}{2}\right)$

(vi) $\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ... + \cos(\alpha + \overline{n-1}\,\beta)$

$= \frac{\sin n\beta/2}{\sin \beta/2} \cdot \cos\left(\frac{\text{Ist angle + last angle}}{2}\right)$

(vii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + ... \infty = \frac{\pi^2}{12}$

(viii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + ... \infty = \frac{\pi^2}{6}$

(ix) $1 + \frac{1}{3^2} + \frac{1}{5^2} + ... + \infty = \frac{\pi^2}{8}$

(x) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + ... \infty = \frac{\pi^2}{24}$

(xi) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

18.4. Summation of Series by Integration

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$, then

$$\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx$$

Rule : Here we proceed as follows :

(i) Express the given series in the form $\Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii) Then the limit is its sum when $n \to \infty$

$$\lim_{n \to \infty} \Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$$

Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx, and $\lim_{n \to \infty} \Sigma$ by the sign of $\int$

(iii) The lower and upper limits of integration will be the values of $\frac{r}{n}$ for the first and last term (or the limit of these values) respectively.

18.5. Gamma Function

The definite integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ is called the **second Eulerian integral** and is denoted by the symbol Γn [read as Gamma n].

Properties of Gamma function

$$\Gamma(n+1) = n\Gamma n; \quad \Gamma 1 = 1; \quad \Gamma(\tfrac{1}{2}) = \Gamma\pi$$

$\Gamma(n+1) = n!$ Provided n is a positive integer.

we have $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

where $m > -1$ and $n > -1$

(This formula is applicable only when the limit is 0 to $\pi/2$).

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If $\int_0^1 \frac{e^t\, dt}{t+1} = a$, then $\int_{b-1}^{b} \frac{e^{-t}\, dt}{t-b-1}$ is equal to

(a) ae^{-b} (b) $-ae^{-b}$

(c) $-be^{-a}$ (d) ae^{b}

2. The value of the integral

$$\int_0^\pi \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin x/2}\, dx \; (n \in N) \text{ is}$$

(a) π (b) 2π

(c) 3π (d) None of these

3. $\int_0^{\pi/3} [\sqrt{3} \tan x]\, dx =$

([.] denotes the greatest integer function)

(a) $5\pi/6$ (b) $\frac{5\pi}{6} - \tan^{-1}(2/\sqrt{3})$

(c) $\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (d) None of these

4. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1}$ is equal to

(a) 0 (b) 1

(c) $-\pi/2$ (d) $\pi/2$

5. The value of $\int_{-1}^{1} [x [1 + \sin \pi x] + 1]\, dx$ is ([.] denotes the greatest integer)
(a) 2 (b) 0
(c) 1 (d) None of these

6. $\int_{-\pi/4}^{n\pi - \pi/4} |\sin x + \cos x|\, dx$ ($n \in N$) is
(a) 0 (b) $2n$
(c) $2\sqrt{2}\, n$ (d) None of these

7. Let $I = \int_0^{\pi/2} \frac{\sin x}{x}\, dx$ then I lies in the interval
(a) (0, 1) (b) [0, 1]
(c) [0, 3/2] (d) [1, 2)

8. For any $t \in R$ and f be a continuous function,
Let $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f(x(2-x))\, dx$
and $I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\, dx$ then $\frac{I_1}{I_2}$ is
(a) 0 (b) 1
(c) 2 (d) 3

9. The value of the integral
$\int_0^{n\pi + t} (|\cos x| + |\sin x|)\, dx$ is
(a) n (b) $2n + \sin t + \cos t$
(c) $\cos t$ (d) $4n + \sin t - \cos t + 1$

10. If $\int_0^{\pi} \log \sin x\, dx = k$, then the value of $\int_0^{\pi/4} \log(1 + \tan x)\, dx$ is
(a) $-\frac{k}{4}$ (b) $\frac{k}{4}$
(c) $-\frac{k}{8}$ (d) $\frac{k}{8}$

11. $\int_{-\pi/4}^{\pi/4} \frac{e^x . \sec^2 x\, dx}{e^{2x} - 1}$ is equal to
(a) 0 (b) 2
(c) e (d) None of these

12. The value of $\int_0^{2\pi} \frac{dx}{e^{\sin x} + 1}$ is
(a) π (b) 0
(c) 2π (d) $\pi/2$

13. $\int_0^{\pi/4} \sin x\, d(x - [x])$ is equal to
(a) 1/2 (b) $1 - \frac{1}{\sqrt{2}}$
(c) 1 (d) None of these

14. The value of the integral
$I = \int_1^{\infty} \frac{(x^2 - 2)}{x^3 \sqrt{(x^2 - 1)}}\, dx$ is
(a) 0 (b) 2/3
(c) 4/3 (d) None of these

15. If $I_1 = \int_0^{3\pi} f(\cos^2 x)\, dx$ and $I_2 = \int_0^{\pi} f(\cos^2 x)\, dx$ then
(a) $I_1 = I_2$ (b) $I_1 = 2I_2$
(c) $I_1 = 5I_2$ (d) None of these

16. If $I = \int_0^{50\pi} \sqrt{(1 - \cos 2x)}\, dx$, then the value of I is
(a) $50\sqrt{2}$ (b) $100\sqrt{2}$
(c) $25\sqrt{2}$ (d) None of these

17. Let $f(x)$ be an odd function in the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$, with a period T. Then $F(x) = \int_a^x f(t)\, dt$ is
(a) periodic with period T
(b) non periodic
(c) periodic with period $2T$
(d) periodic with period aT

18. If $\int_0^{10} f(x)\, dx = 5$, then $\sum_{k=1}^{10} \int_0^1 f(k - 1 + x)\, dx$ is
(a) 50 (b) 10
(c) 5 (d) None of these

19. The value of the integral $\left| \int_0^{2\pi} [2 \sin x]\, dx \right|$ is ([.] denotes the greatest integer function)
(a) π (b) 2π
(c) 3π (d) 4π

20. Let $f(x) =$ minimum $\left(|x|, 1 - |x|, \frac{1}{4} \right)$, $\forall x \in R$, then the value of $\int_{-1}^{1} f(x)\, dx$ is equal to
(a) $\frac{1}{32}$ (b) $\frac{3}{8}$

(c) $\frac{3}{32}$ (d) none of these

21. If $\int_{1/2}^{2} \frac{1}{x} \text{cosec}^{101}\left(x - \frac{1}{x}\right) dx = k$ then the value of k is
(a) 1 (b) 1/2
(c) 0 (d) 1/101

22. If $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ and $\int_0^{\infty} e^{-ax^2} dx, a > 0$ is
(a) $\frac{\sqrt{\pi}}{2}$ (b) $\frac{\sqrt{\pi}}{2a}$
(c) $2\frac{\sqrt{\pi}}{a}$ (d) $\frac{1}{2}\sqrt{\frac{\pi}{a}}$

23. If $\int_0^1 e^{x^2}(x - \alpha)\, dx = 0$, then
(a) $1 < \alpha < 2$ (b) $\alpha < 0$
(c) $0 < \alpha < 1$ (d) $\alpha = 0$

24. If $[x]$ denotes the greatest integer less than or equal to x then $\int_0^{\infty}\left[\frac{2}{e^x}\right] dx$ is equal to
(a) $\log_e 2$ (b) e^2
(c) 0 (d) $\frac{2}{e}$

25. If $\int_0^x f(t)\, dt = x + \int_x^1 t f(t)\, dt$ then the value of $f(1)$ is
(a) 1/2 (b) 0
(c) 1 (d) −1/2

26. The value of $\int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x}\, dx$ is equal to
(a) $(\sqrt{2} - 1)\pi$ (b) $(\sqrt{2} + 1)\pi$
(c) π (d) None of these

27. Let f be a positive function. If

$$I_1 = \int_{1-k}^{k} x f\{x(1-x)\}\, dx, I_2 = \int_{1-k}^{k} f\{x(1-x)\}\, dx$$

where $2k - 1 > 0$, then $I_1 : I_2$ is equal to
(a) 2 : 1 (b) k : 1
(c) 1 : 2 (d) 1 : 1

28. Let $f(x) = \frac{1}{2} a_0 + \sum_{i=1}^{n} a_i \cos(ix) + \sum_{j=1}^{n} b_j \sin(jx)$ then $\int_{-\pi}^{\pi} f(x) \cos kx\, dx$ is equal to
(a) a_k (b) b_k
(c) πa_k (d) πb_k

29. The value of the definite integral $\int_0^1 \frac{x\, dx}{(x^2 + 16)}$ lies in the interval $[a, b]$. Then smallest such interval is
(a) $\left[0, \frac{1}{17}\right]$ (b) [0, 1]
(c) $\left[0, \frac{1}{27}\right]$ (d) None of these

30. Value of $\int_2^3 \frac{dx}{\sqrt{(1 + x^3)}}$ is
(a) less than 1
(b) greater than 2
(c) lies between 3 and 4
(d) None of these

31. Suppose for every integer n, $\int_n^{n+1} f(x)\, dx = n^2$ the value of $\int_{-2}^{4} f(x)\, dx$ is
(a) 16 (b) 14
(c) 19 (d) None of these

32. The value of

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt \text{ is}$$

(a) $\pi/2$ (b) 1
(c) $\pi/4$ (d) None of these

33. If $I = \int_0^1 \cos\left(2 \cot^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right) dx$ then
(a) $I > \frac{1}{2}$ (b) $I = -\frac{1}{2}$
(c) $0 < I < \frac{1}{2}$ (d) None of these

34. The value of $\int_0^2 [x^2 - 1]\, dx$, where $[x]$ denotes the greatest integer function, is given by
(a) $3 - \sqrt{3} - \sqrt{2}$ (b) 2
(c) 1 (d) None of these

35. Let $\int_0^x \left(\frac{bt \cos 4t - a \sin 4t}{t^2}\right) dt = \frac{a \sin 4x}{x}$ then a and b are given by

(a) $a = 1/4, b = 1$ (b) $a = 2, b = 2$
(c) $a = -1, b = 4$ (d) $a = 2, b = 4$

36. If $f(x) = \cos x - \int_0^x (x - t) f(t)\, dt,$ then $f''(x) + f(x)$ equals
(a) $-\cos x$ (b) 0
(c) $\int_0^x (x - t) f(t)\, dt$ (d) $-\int_0^{-x} (x - t) f(t)\, dt$

37. Let $f(x) = \max \{x + |x|, x - [x]\}$, where $[x]$ denotes the greatest integer $< x$. Then $\int_{-2}^{2} f(x)\, dx$ is equal to
(a) 3 (b) 2
(c) 1 (d) None of these

38. The value of $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$, where $[x]$ denotes the greatest integer $< x$, is
(a) 1 (b) 0
(c) $4 - \sin 4$ (d) None of these

39. The value of $\int_{-1}^{1} \max \{2 - x, 2, 1 + x\}\, dx$ is
(a) 4 (b) 9/2
(c) 2 (d) None of these

40. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and g be the function satisfying $f(x) + g(x) = x^2$ the value of the integral $\int_0^1 f(x)\, g(x)\, dx$ is
(a) $\frac{1}{4}(e - 7)$ (b) $\frac{1}{4}(e - 2)$
(c) $\frac{1}{2}(e - 3)$ (d) None of these

41. The value of $\int_0^{\pi} \left(\sum_{r=0}^{3} a_r \cos^{3-r} x \sin^r x \right) dx$ depends on
(a) a_0 and a_2 (b) a_1 and a_2
(c) a_0 and a_3 (d) a_1 and a_3

42. Let $f : R \to R$ such that $f(x + 2y) = f(x) + f(2y) + 4xy\ \forall\ x, y \in R$ and $f'(0) = 0$. If
$$I_1 = \int_0^1 f(x)\, dx,\ I_2 = \int_{-1}^{0} f(x)\, dx \text{ and}$$
$$I_3 = \int_{1/2}^{2} f(x)\, dx \text{ then}$$
(a) $I_1 = I_2 > I_3$ (b) $I_1 > I_2 > I_3$
(c) $I_1 = I_2 < I_3$ (d) $I_1 < I_2 < I_3$

43. If $\int_a^b \frac{x^n}{x^n + (16 - x)^n} dx = 6$ then
(a) $a = 4, b = 12, n \in R$
(b) $a = 2, b = 14, n \in R$
(c) $a = -4, b = 20, n \in R$
(d) $a = 2, b = 8, n \in R$

44. If $f(x)$ and $g(x)$ are continuous functions, then
$$\int_{\ln \lambda}^{\ln 1/\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x) - f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x) + g(-x)]} dx \text{ is}$$
(a) depend on λ (b) a non-zero constant
(c) zero (d) none of these

45. The value of $\int_0^{\pi/2} \frac{1 + 2\cos x}{(2 + \cos x)^2} dx$ is
(a) $-\frac{1}{2}$ (b) 2
(c) $\frac{1}{2}$ (d) None of these

46. The value of $\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$ is
(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$
(c) π^2 (d) $2\pi^2$

47. Consider the integrals
$$I_1 = \int_0^1 e^{-x} \cos^2 x\, dx,\ I_2 = \int_0^1 e^{-x^2} \cos^2 x\, dx,$$
$$I_3 = \int_0^1 e^{-x^2} dx \text{ and } I_4 = \int_0^1 e^{-x^2/2} dx$$
let I be the greatest integral among I_1, I_2, I_3, I_4 then
(a) $I = I_1$ (b) $I = I_2$
(c) $I = I_3$ (d) $I = I_4$

48. If $f(x) = \int_2^{x^2} \frac{(\sin^{-1} \sqrt{t})^2}{\sqrt{t}} dt$ then the value of $(1 - x^2) \{f''(x)\}^2 - 2f'(x)$ at $x = \frac{1}{\sqrt{2}}$ is
(a) $2 - \pi$ (b) $3 + \pi$
(c) $4 - \pi$ (d) None of these

49. If for $x \neq 0$ $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$ where $a \neq b$

then $\int_1^2 x f(x)\, dx =$

(a) $\frac{b-9a}{9(a^2-b^2)}$ (b) $\frac{b-9a}{b(a^2-b^2)}$

(c) $\frac{b-9a}{6(a^2-b^2)}$ (d) None of these

50. Let f and g be two continuous functions. Then

$\int_{-\pi/2}^{\pi/2} \{f(x) + f(-x)\}\{g(x) - g(-x)\}\, dx$

is equal to

(a) π (b) 1

(c) -1 (d) 0

51. Let $f(x)$ be a continuous function such that $f(a-x) + f(x) = 0$ for all $x \in [0, a]$,

Then $\int_0^a \frac{dx}{1 + e^{f(x)}}$ is equal to

(a) a (b) $\frac{a}{2}$

(c) $f(a)$ (d) $\frac{1}{2} f(a)$

52. The equation

$\int_{-\pi/4}^{\pi/4} \left(a \,|\sin x|\, + \frac{b \sin x}{1 + \cos x} + c \right) dx = 0,$

where a, b, c are constants, gives a relation between

(a) a, b and c (b) a and c

(c) a and b (d) b and c

53. The value of $\int_0^{16\pi/3} |\sin x|\, dx$ is

(a) 17/2 (b) 19/2

(c) 21/2 (d) None of these

54. The value of $\int_0^{1000} e^{x - [x]}\, dx$, is

([.] denotes the greatest integer function)

(a) $1000\, e$ (b) $1000\,(e-1)$

(c) $1001\,(e-1)$ (d) None of these

55. If $f'''(x) = k$ in $[0, a]$ then

$\int_0^a f(x)\, dx - \left\{ x f(x) - \frac{x^2}{2!} f'(x) + \frac{x^3}{3!} f''(x) \right\}_0^a =$

(a) $-ka^4/12$ (b) $ka^4/24$

(c) $-ka^4/24$ (d) None of these

56. The value of $\int_0^{n^2} [\sqrt{x}]\, dx$, ([.] denotes the greatest integer function), $n \in N$ is

(a) $\left\{ \frac{n(n+1)}{2} \right\}^2$

(b) $\frac{1}{6} n(n-1)(4n+1)$

(c) Σn^2

(d) $\frac{n(n+1)(n+2)}{6}$

57. $\lim_{n \to \infty} \frac{(n!)^{1/n}}{n}$ equals

(a) e (b) e^{-1}

(c) 1 (d) None of these

58. If $[x]$ stands for the greatest integer function, the value of $\int_4^{10} \frac{[x^2]\, dx}{[x^2 - 28x + 196] + [x^2]}$ is

(a) 0 (b) 1

(c) 3 (d) none of these

59. $\int_{\cos \cos^{-1} \alpha}^{\sin \sin^{-1} \beta} \begin{vmatrix} \frac{\cos(\cos^{-1} x)}{\sin(\sin^{-1} x)} \end{vmatrix} dx$ is equal to

(a) 1 (b) 0

(c) $\beta - \alpha$ (d) None of these

60. If $na = 1$ always and $n \to \infty$ then the value of $\Pi\, \{1 + (ra)^2\}^{1/r}$ is

(a) 1 (b) $e^{\pi^2/8}$

(c) $e^{\pi^2/24}$ (d) $e^{-\pi^2/12}$

MULTIPLE CHOICE –II

Each question in this part has one or more than one correct answer (s). For each quesion, write the letters a, b, c, d corresponding to the correct answer (s).

61. The value of $\int_0^{100} [\tan^{-1} x]\, dx$ is

(when [.] denotes the greatest integer function)

(a) 100 (b) $100 - \tan^{-1} 1$

(c) $100 - \tan 1$ (d) None of these

62. If x satisfies the equation
$$x^2\left(\int_0^1 \frac{dt}{t^2+2t\cos\alpha+1}\right)-x\left(\int_{-3}^{3}\frac{t^3\sin 2t}{t^2+1}\,dt\right)-2=0$$
$(0<\alpha<\pi)$, then the value of x is

(a) $2\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$ (b) $-2\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$
(c) $4\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$ (d) None of these

63. The value of the integral $\int_a^b \frac{|x|}{x}\,dx,\ a<b$ is

(a) $b-a$ (b) $a-b$
(c) $b+a$ (d) None of these

64. If $f(x)$ satisfies the requirements of Rolle's theorem in [1, 2] and $f'(x)$ is continuous in [1, 2], then $\int_1^2 f'(x)\,dx$ is equal to

(a) 0 (b) 1
(c) 3 (d) –1

65. The value of the integral $\int_0^1 e^{x^2}\,dx$ is

(a) less than e (b) greater than e
(c) less than 1 (d) greater than 1

66. If $I_1=\int_0^1 2^{x^2}\,dx,\ I_2=\int_0^1 2^{x^3}\,dx,\ I_3=\int_1^2 2^{x^2}\,dx$ and $I_4=\int_1^2 2^{x^3}\,dx$ then

(a) $I_1>I_2$ (b) $I_2>I_1$
(c) $I_3>I_4$ (d) $I_4>I_3$

67. The absolute value of $\int_{10}^{19}\frac{\sin x}{1+x^8}\,dx$ is

(a) less than 10^{-7} (b) more than 10^{-7}
(c) less than 10^{-6} (d) None of these

68. If $I=\int_3^4 \frac{dx}{\sqrt[3]{(\log x)}}$, then

(a) $I<1$ (b) $I>1$
(c) $I>0{\cdot}92$ (d) $I<0{\cdot}92$

69. If $I_1=\int_x^1 \frac{dt}{1+t^2}$ and $I_2=\int_1^{1/x}\frac{dt}{1+t^2}$ for $x>0$, then

(a) $I_1=I_2$ (b) $I_1>I_2$
(c) $I_2>I_1$ (d) $I_2=\cot^{-1}x-\pi/4$

70. The value of $\int_a^{a+\pi/2}(\sin^4 x+\cos^4 x)\,dx$ is

(a) independent of a (b) $a\left(\frac{\pi}{2}\right)^2$
(c) $3\pi/8$ (d) $\frac{3}{8}\pi a^2$

71. The value α in the interval $[-\pi, 0]$ staisfying $\sin\alpha+\int_\alpha^{2\alpha}\cos 2x\,dx=0$ is

(a) $-\pi/2$ (b) $-\pi$
(c) $-\pi/3$ (d) 0

72. The value of $\int_{-2}^{2}\min(x-[x],-x-[-x])\,dx$ is ([.] denotes the greatest integer function)

(a) 0 (b) 1
(c) 2 (d) None of these

73. The value of $\int_0^2 [x^2-x+1]\,dx$, (where $[x]$ denotes the greatest integer function) is given by

(a) $\frac{5-\sqrt{5}}{2}$ (b) $\frac{6-\sqrt{5}}{2}$
(c) $\frac{7-\sqrt{5}}{2}$ (d) $\frac{8-\sqrt{5}}{2}$

74. If $I=\int_0^1\sqrt{(1+x^3)}\,dx$ then

(a) $I<1$ (b) $I\neq\sqrt{5}/2$
(c) $I<\sqrt{7}/2$ (d) None of these

75. The value of $\lim_{n\to\infty}\left(\frac{1^k+2^k+\ldots+n^k}{n^{k+1}}\right)$ is

(a) $\frac{1}{k+2}$ (b) $\frac{1}{k+1}$
(c) $\frac{2}{k+3}$ (d) 0

76. The value of the integral $\int_{1/e}^{e}|\ln x|\,dx$ is

(a) $1-1/e$ (b) $3(1-1/e)$
(c) $e^{-1}-1$ (d) None of these

77. $f(x)$ is continuous periodic function with period T, then the integral $I=\int_a^{a+T}f(x)\,dx$ is

(a) equal to $2a$ (b) equal to $3a$
(c) independent of a (d) None of these

78. $\lim_{n \to \infty} \frac{1}{n} \sum_{r=n+1}^{2n} \log_e \left(1 + \frac{r}{n}\right)$ equals

(a) $\log\left(\frac{27}{4e}\right)$ (b) $\log\left(\frac{27}{e^2}\right)$

(c) $\log\left(\frac{4}{e}\right)$ (d) None of these

79. The maximum and minimum value of the integral $\int_0^{\pi/2} \frac{dx}{(1+\sin^2 x)}$ are

(a) $\pi/4$ (b) π

(c) $\pi/2$ (d) $3\pi/4$

80. Given that n is odd and m is even integer. The value of $\int_0^{\pi} \cos mx \sin nx\, dx$ is

(a) $\frac{2m}{n^2 - m^2}$ (b) $\frac{2n}{n^2 - m^2}$

(c) $\frac{m^2 + n^2}{n^2 - m^2}$ (d) None of these

81. Let $f(a) > 0$, and let $f(x)$ be a non decreasing continuous function in $[a, b]$, Then $\frac{1}{b-a}\int_a^b f(x)\, dx$ has the

(a) maximum value $f(b)$

(b) minimum value $f(a)$

(c) maximum value $bf(b)$

(d) minimum value $\frac{f(a)}{b-a}$

82. $\int \frac{(x^4 - x)^{1/4}}{x^5} dx$ is equal to

(a) $\frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{5/4} + c$

(b) $\frac{4}{5}\left(1 - \frac{1}{x^3}\right)^{5/4} + c$

(c) $\frac{4}{15}\left(1 + \frac{1}{x^3}\right)^{5/4} + c$

(d) None of these

83. $\int_{a/4}^{3a/4} \frac{\sqrt{x}}{\sqrt{(a-x} + \sqrt{x}} dx$ is equal to

(a) $\frac{a}{2}$ (b) a

(c) $-a$ (d) None of these

84. The points of extremum of $\int_0^x \frac{t^2 - 5t + 4}{2 + e^t} dt$ are

(a) $x = -2$ (b) $x = 1$

(c) $x = 0$ (d) $x = -1$

85. The values of α which satisfy $\int_{\pi/2}^{\alpha} \sin x\, dx = \sin 2\alpha$ ($\alpha \in [0, 2\pi]$) are equal to

(a) $\pi/2$ (b) $3\pi/2$

(c) $7\pi/6$ (d) $11\pi/6$

86. $\int_{-1}^{10} Sgn\,(x - [x])\, dx$ equals [Here [.] denotes greatest integer function]

(a) 10 (b) 11

(c) 9 (d) 11/2

87. If $f(x) = \int \frac{\log x}{1+x} dx$, then $f(x) + f\left(\frac{1}{x}\right) =$

(a) $(\log x)^2$ (b) $\frac{1}{2}(\log x)^2$

(c) $\frac{1}{2}(x \log x)^2$ (d) None of these

88. The value of $\int_0^2 \left|\cos\left(\frac{\pi x}{2}\right)\right| dx$ is

(a) 2π (b) $\pi/2$

(c) $3/4\,\pi$ (d) 4π

89. The number of positive continuous functions $f(x)$ defined in [0, 1] for which

(a) one (b) infinite

(c) two (d) zero

90. If $f'(x) = f(x) + \int_0^1 f(x)\, dx$ and given $f(0) = 1$ then $f(x) =$

(a) $\frac{e^x}{2-e} + \left(\frac{1+e}{1-e}\right)$

(b) $\frac{2e^x}{3-e} + \left(\frac{1-e}{1+e}\right)$

(c) $\frac{e^x}{2-e}$

(d) $\frac{2e^x}{3-e}$

Practice Test

M.M. : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. The value of the integral $\int_0^1 \frac{x^\alpha - 1}{\log \alpha}\, dx$, is

(a) $\log \alpha$ (b) $2 \log (\alpha + 1)$
(c) $3 \log \alpha$ (d) None of these

2. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, then

(a) $I_n + I_{n-2} = \frac{1}{n-1}$

(b) $I_{n+1} + I_{n-1} = \frac{1}{n}$

(c) $\frac{1}{n+1} < 2I_n < \frac{1}{n-1}$, where $n > 1$ is a natural number

(d) $(I_{n+2} + I_n)(n+1) = 1$

3. The value of $\int_\pi^{2\pi} [2 \sin x]\, dx$, (when [.] represents the greatest integer function) is

(a) $-\frac{5\pi}{3}$ (b) $-\pi$
(c) $\frac{5\pi}{3}$ (d) -2π

4. $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^{1/a}\left\{n^{a-\frac{1}{a}} + k^{a-\frac{1}{a}}\right\}}{n^{a+1}} =$

(a) 1 (b) 2
(c) 3 (d) None of these

5. Let $f(x) = \min(\{x\}, \{-x\})\ \forall x \in R$, where $\{x\}$ denotes the fractional part of x, then $\int_{-100}^{100} f(x)\, dx$ is equal to

(a) 50 (b) 100
(c) 200 (d) None of these

6. The value of $\int_{-\pi/2}^{199\pi/2} \sqrt{(1 + \cos 2x)}\, dx$ is

(a) $50\sqrt{2}$ (b) $100\sqrt{2}$
(c) $150\sqrt{2}$ (d) $200\sqrt{2}$

7. The value of

$\int_0^{\pi/4} (\tan^n x + \tan^{n-2} x)\, d(x - [x])$,

(where [.] denotes the greatest integer function) is

(a) $\frac{1}{n-1}$ (b) $\frac{1}{n+1}$
(c) $\frac{2}{n-1}$ (d) None of these

8. If $\int_{-1}^{4} f(x)\, dx = 4$ and $\int_2^4 (3 - f(x))\, dx = 7$ the value of $\int_2^{-1} f(x)\, dx$ is

(a) 2 (b) -3
(c) -5 (d) None of these

9. If $I_1 = \int_0^x e^{zx} e^{-z^2}\, dz$ and $I_2 = \int_0^x e^{-z^2/4}\, dz$ then

(a) $I_1 = e^{x} I_2$ (b) $I_1 = e^{x^2} I_2$
(b) $I_1 = e^{x^2/2} I_2$ (d) None of these

10. If $z = x + 3i$ then value of $\int_2^4 \left[\arg \left|\frac{z-i}{z+i}\right|\right] dx$, where [.] denotes the greatest integer function, is

(a) $3\sqrt{2}$ (b) $6\sqrt{3}$
(c) $\sqrt{6}$ (d) none

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice–I

1. (b)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (c)
7. (d)	**8.** (b)	**9.** (d)	**10.** (c)	**11.** (a)	**12.** (a)
13. (b)	**14.** (a)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (c)
19. (a)	**20.** (b)	**21.** (c)	**22.** (d)	**23.** (c)	**24.** (b)
25. (a)	**26.** (a)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (a)
31. (c)	**32.** (c)	**33.** (b)	**34.** (a)	**35.** (a)	**36.** (a)
37. (d)	**38.** (b)	**39.** (b)	**40.** (d)	**41.** (d)	**42.** (c)
43. (b)	**44.** (c)	**45.** (c)	**46.** (c)	**47.** (d)	**48.** (d)
49. (d)	**50.** (d)	**51.** (b)	**52.** (b)	**53.** (c)	**54.** (b)
55. (c)	**56.** (b)	**57.** (b)	**58.** (c)	**59.** (c)	**60.** (c)

Multiple Choice–II

61. (c)	**62.** (a), (b)	**63.** (a), (b), (c)	**64.** (a)	**65.** (a), (d)	**66.** (a), (d)
67. (a), (c)	**68.** (a), (c)	**69.** (a), (d)	**70.** (a), (c)	**71.** (b), (c), (d)	**72.** (b)
73. (c)	**74.** (a), (c)	**75.** (b)	**76.** (b)	**77.** (c)	**78.** (a)
79. (a), (c)	**80.** (b)	**81.** (a), (b)	**82.** (a)	**83.** (a)	
84. (a,) (b), (c), (d)		**85.** (a), (b), (c), (d)		**86.** (b)	**87.** (b)
88. (d)	**89.** (d)	**90.** (b)			

Practice Test

1. (d)	**2.** (a), (b), (c), (d)	**3.** (a)	**4.** (a)	**5.** (a)
6. (d)	**7.** (a)	**8.** (c)	**9.** (d)	**10.** (d)

19

AREAS

§ 19.1 The area bounded by the continuous curve $y = f(x)$, Fig. 1 the axis of x and the ordinates $x = a$ and $x = b$ (where $b > a$) is given by

$$A = \int_a^b f(x)\, dx = \int_a^b y\, dx$$

§ 19.2 The area bounded by the continuous curve $x = g(y)$, (Fig. 2.) the axis of y and the abscissae $y = c$ and $y = d$ (where $d > c$) is given by

$$A = \int_c^d g(y)\, dy = \int_c^d x dy$$

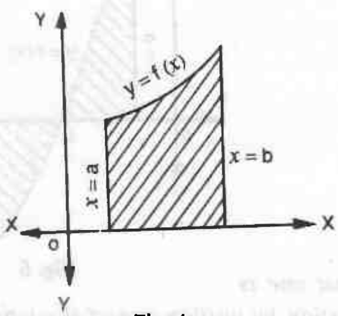


Fig. 1.

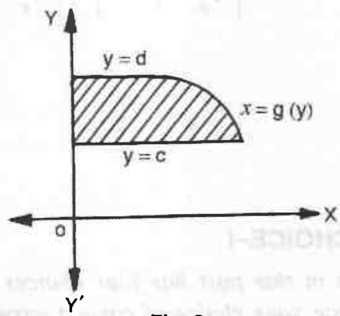


Fig. 2.

§ 19.3 The area bounded by the straight lines $x = a$, $x = b$ $(a < b)$ and the curves $y = f(x)$ and $y = g(x)$, (Fig. 3) provided $f(x) < g(x)$ $(a < x < b)$, is given by

$$A = \int_a^b [g(x) - f(x)]\, dx$$

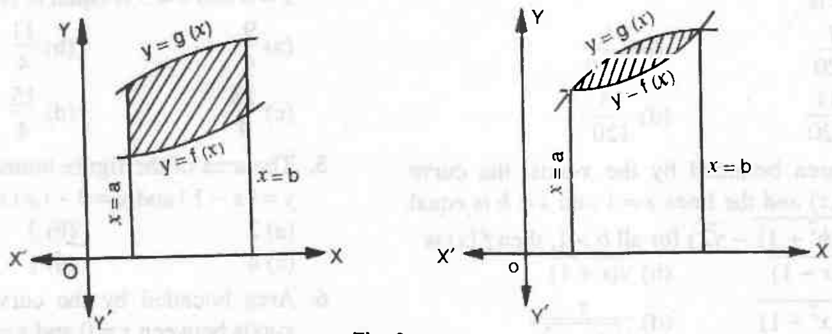


Fig. 3.

Note : If some part of curves lies below the x-axis, then its area is negative but area cannot be negative. Therefore we take its modulus.

(i) If the curve crosses the x-axis in two points (*i.e.*, c, d), Fig. 4 then the area between the curve $y = f(x)$ on the x-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = \left| \int_a^c f(x)\,dx \right| + \left| \int_c^d f(x)\,dx \right| + \left| \int_d^b f(x)\,dx \right|$$

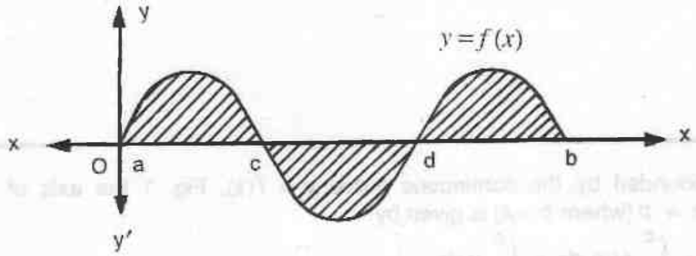


Fig. 4.

(ii) If the curve cross the x-axis at c, then the area bounded by the curve $y = f(x)$ and the ordinates $x = a$ and $x = b$ $(b > a)$ is given by (Fig. 5)

$$A = \left| \int_a^c f(x)\,dx \right| + \left| \int_c^b f(x)\,dx \right|$$

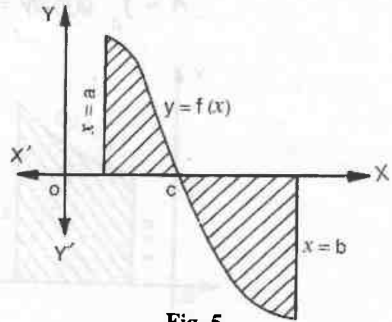


Fig. 5

MULTIPLE CHOICE–I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate :

1. The area between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of two minima of the curve is

(a) $\frac{7}{120}$ (b) $\frac{9}{120}$

(c) $\frac{11}{120}$ (d) $\frac{13}{120}$

2. The area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = 1$ and $x = b$ is equal to $(\sqrt{(b^2 + 1)} - \sqrt{2})$ for all $b > 1$, then $f(x)$ is

(a) $\sqrt{(x - 1)}$ (b) $\sqrt{(x + 1)}$

(c) $\sqrt{(x^2 + 1)}$ (d) $\frac{x}{\sqrt{(1 + x^2)}}$

3. The area of the region bounded by $1 - y^2 = |x|$ and $|x| + |y| = 1$ is

(a) 1/3 (b) 2/3

(c) 4/3 (d) 1

4. Area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ and x-axis lying between the ordinates $x = 0$ and $x = 3$ is equal to (in square units)

(a) $\frac{9}{4}$ (b) $\frac{11}{4}$

(c) $\frac{13}{4}$ (d) $\frac{15}{4}$

5. The area of the figure bounded by the curves $y = |x - 1|$ and $y = 3 - |x|$ is

(a) 2 (b) 3

(c) 4 (d) 1

6. Area bounded by the curve $y = x \sin x$ and x-axis between $x = 0$ and $x = 2\pi$ is

(a) 2π (b) 3π

(c) 4π (d) 5π

7. Let $f(x) = \min [x + 1, \sqrt{(1 - x)}]$, Then area bounded by $f(x)$ and x-axis is

(a) $\frac{1}{6}$ (b) $\frac{5}{6}$

(c) $\frac{7}{6}$ (d) $\frac{11}{6}$

8. The area bounded by the graph $y = |[x-3]|$, the x-axis and the lines $x = -2$ and $x = 3$ is ([.] denotes the greatest integer function)

(a) 7 sq. units (b) 15 sq. units

(c) 21 sq. units (d) 28. sq. units

9. The value of c for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, the straight lines $x = 1$ and $x = c$ and the x-axis is equal to 16/3 is

(a) 2 (b) $\sqrt{8-\sqrt{17}}$

(c) 3 (d) –1

10. Area bounded by the curves $y = \left[\frac{x^2}{64} + 2\right]$, $y = x - 1$ and $x = 0$ above x-axis is ([.] denotes the greatest integer function)

(a) 2 (b) 3

(c) 4 (d) None of these

11. The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$. If the curve passes through the point (1, 2), then the area of the region bounded by the curve, the x-axis and the line $x = 1$ is

(a) 5/6 (b) 6/5

(c) 1/6 (d) 6

12. The area of the region bounded by the curve $a^4y^2 = (2a - x)x^5$ is to that of the circle whose radius is a, is given by the ratio

(a) 4 : 5 (b) 5 : 8

(c) 2 : 3 (d) 3 : 2

13. The area bounded by $y = xe^{|x|}$ and lines $|x| = 1$, $y = 0$ is

(a) 4 (b) 6

(c) 1 (d) 2

14. The area of the figure bounded by $f(x) = \sin x$, $g(x) = \cos x$ in the first quadrant is

(a) $2(\sqrt{2} - 1)$ (b) $\sqrt{3} + 1$

(c) $2(\sqrt{3} - 1)$ (d) None of these

15. The area of the figure bounded by two branches of the curve $(y - x)^2 = x^3$ and the straight line $x = 1$ is

(a) 1/3 sq. units (b) 4/5 sq. units

(c) 5/4 sq. units (d) 3 sq. units

16. If the area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = c$ and $x = d$ is independent of d, $\forall\, d > c$ (c is a constant), then f is

(a) the identity function

(b) the zero function

(c) a non zero constant function

(d) None of these

17. The area bounded by the curve $y = |x| - 1$ and $y = -|x| + 1$ is :

(a) 1 (b) 2

(c) $2\sqrt{2}$ (d) $4\sqrt{2}$

18. The area bounded by $y = x^2$, $y = [x + 1]$, $x < 1$ and the y-axis is ([.] denotes the greatest integer function)

(a) 1/3 (b) 2/3

(c) 1 (d) 7/3

19. Let $f(x)$ be a continuous function such that the area bounded by the curve $y = f(x)$, the x-axis and the two ordinates $x = 0$ and $x = a$ is $\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$, then $f(\pi/2)$ is

(a) $\frac{1}{2}$ (b) $\frac{\pi^2}{8} + \frac{\pi}{4}$

(c) $\frac{\pi + 1}{2}$ (d) None of these

20. The area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$, the axis of abscissas and two ordinates corresponding to the points of minimum of the function $y(x)$ is

(a) 10/3 (b) 27/10

(c) 21/10 (d) None of these

21. The are bounded by the curves $y = ln\, x$, $y = \ln|x|$, $y = |ln\, x|$ and $y = |ln|x||$ is

(a) 5 sq. units (b) 2 sq. units

(c) 4 sq. units (d) None of these

22. The area bounded by the curve $y = x^2 + 2x + 1$ and tangent at (1, 4) and y-axis is

(a) $\frac{2}{3}$ sq. units (b) $\frac{1}{3}$ sq. units

(c) 2 sq. units (d) None of these

23. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the co-ordinate axes, lies in the first quadrant. If its area is 2, then the value of 'b' is

(a) –3 (b) –2

(c) –1 (d) 0

24. The area bounded by $y = 2 - |2 - x|$, $y = \frac{3}{|x|}$ is

(a) $\frac{5 - 4\ ln\ 2}{3}$

(b) $\frac{2 - ln\ 3}{2}$

(c) $\frac{4 - 3\ ln\ 3}{2}$

(d) None of these

25. If $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$ then the area of the region bounded by the curves $y = f(x)$, x-axis, y-axis and $x = 2\pi$ is

(a) $\left(\frac{5\pi}{12} + 3\right)$ sq. units

(b) $\left(\frac{5\pi}{12} + \sqrt{2}\right)$ sq. units

(c) $\left(\frac{5\pi}{12} + \sqrt{3}\right)$ sq. units

(d) $\left(\frac{5\pi}{12} + \sqrt{2} + \sqrt{3}\right)$ sq. units

Practice Test

M.M : 10 Time : 15 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. Area of the region bounded by the curve $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$ is given by

(a) $e - e^{-1} + 2$ (b) $e - e^{-1} - 2$

(c) $e + e^{-1} - 2$ (d) None of these

2. The area bounded by the curve $y = f(x)$, the x-axis and the ordinate $x = 1$ and $x = b$ is $(b - 1)\cos(3b + 4)$. Then $f(x)$ is given by

(a) $(x - 1)\sin(3x + 4)$

(b) $3(x - 1)\sin(3x + 4) + \cos(3x + 4)$

(c) $\cos(3x + 4) - 3(x - 1)\sin(3x + 4)$

(d) None of these

3. Let $f(x) = \begin{cases} x^2 : & x \le 0 \\ x : & x > 0 \end{cases}$ Area bounded by the curve $y = f(x)$, $y = 0$ and $x = \pm 3a$ is $9a/2$. Then $a =$

(a) 1/2 (b) –1/2

(c) 0 (d) None of these

4. The area bounded by the curves

$$|x| + |y| > 1 \text{ and } x^2 + y^2 < 1 \text{ is}$$

(a) 2 sq. units (b) π sq. units

(c) $(\pi - 2)$ sq. units (d) $(\pi + 2)$ sq. units

5. If A_n is the area bounded by $y = (1 - x^2)^n$ and coordinates axes, $n \in N$, then

(a) $A_n = A_{n-1}$

(b) $A_n < A_{n-1}$

(c) $A_n > A_{n-1}$

(d) $A_n = 2A_{n-1}$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a)	**2.** (d)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (c)
7. (c)	**8.** (b)	**9.** (d)	**10.** (c)	**11.** (a)	**12.** (b)
13. (d)	**14.** (a)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (b)
19. (a)	**20.** (b)	**21.** (c)	**22.** (b)	**23.** (a)	**24.** (c)
25. (d)					

Practice Test

1. (c)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (b)

20

DIFFERENTIAL EQUATIONS

§ 20.1. Variable Separable :

If the differential Equation of the form

$$f_1(x)\,dx = f_2(y)\,dy \qquad ...(1)$$

where f_1 and f_2 being functions of x and y only. then we say that the variables are separable in the differential equation.

thus, integrating both sides of (1), we get its solution as

$$\int f_1(x)\,dx = \int f_2(y)\,dy + c$$

where c is an arbitrary constant.

Method of Substitution :

If the differential equation is not in the form of variable separable but after proper substitution the equation reduces in variable separable form in the new variable.

§ 20.2. Homogeneous Differential Equations :

A differential equation of the form

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}, \text{ where } f_1(x, y) \text{ and } f_2(x, y)$$

are homogeneous functions of x and y of the same degree, is called a **homogeneous equation.**

Working Rule : To get the solution of a homogeneous differential equation, we follow the following procedure :

(i) Put $y = vx$ so that $v + x\frac{dv}{dx} = \frac{dy}{dx}$.

(ii) The equation thus obtained will be of the form in which variables are separable.

(iii) After integration replace v by y/x and get the general solution.

Equations Reducible to Homogenous Form :

A differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1x + b_1y + c_1}$$

Case (i) If $\frac{a}{a_1} = \frac{b}{b_1} = m$ (say)

then $\frac{dy}{dx} = \frac{m(a_1x + b_1y) + c}{(a_1x + b_1y + c_1)}$ where m is any number.

In such case the substitution

$$a_1x + b_1y = v, \text{ so that } a_1 + b_1\frac{dy}{dx} = \frac{dv}{dx}$$

transform the differential equation to the form

$$\frac{1}{b_1}\left(\frac{dv}{dx} - a_1\right) = \frac{mv + c}{v + c_1}$$

or

$$\frac{dv}{dx} = a_1 + b_1\left(\frac{mv + c}{v + c_1}\right)$$

which is a differential equation in variables separable form and it can easily be solved.

Case (ii) : If $\frac{a}{a_1} \neq \frac{b}{b_1}$

then put $x = X + h,\ y = Y + k$, so that

$$\frac{dY}{dX} = \frac{dy}{dx}$$

transform the differential equation to the form

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a_1X + b_1Y + (a_1h + b_1k + c_1)}$$

Now, choose h and k such that

$$ah + bk + c = 0$$
$$a_1h + b_1k + c_1 = 0$$

then for these values of h and k, the equation becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a_1X + b_1Y}$$

This is a homogeneous equation which can be solved by putting $Y = vX$. finally by replacing X by $(x - h)$ and Y by $(y - k)$ we shall get the solution in original variables x and y.

§ 20.3. Linear Differential Equation :

A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone or constants is known as linear differential equation and its solution is given by

$$ye^{\int Pdx} = \int Q(e^{\int Pdx})\, dx + c$$

Note : If the differential equation can be represented in the form $\frac{dx}{dy} + Px = Q$. where P and Q are functions of y alone or constants and its solution is given by

$$x\, e^{\int Pdy} = \int Q(e^{\int Pdy})\, dy + c$$

Equations Reducible to the Linear form :

(i) **Bernoullis' Equation :** An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone or constants and n is constant, other than 0 and 1, is called a **Bernoulli's equation.**

Dividing by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$$

Now put $y^{-n+1} = v$, so that $(-n+1)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$,

we get

$$\frac{dv}{dx} + (1 - n)\, Pv = (1 - n)\, Q.$$

which is linear differential equation.

(ii) If the equation of the form

$$\frac{dy}{dx} + P\phi(y) = Q\,\psi(y),$$

where P and Q are functions of x alone or constants.

Dividing by $\psi(y)$, we get

$$\frac{1}{\psi(y)}\frac{dy}{dx} + \frac{\phi(y)}{\psi(y)}P = Q$$

Now put $\frac{\phi(y)}{\psi(y)} = v$, so that $\frac{d}{dx}\left\{\frac{\phi(y)}{\psi(y)}\right\} = \frac{dv}{dx}$

or $$\frac{dv}{dx} = k \cdot \frac{1}{\psi(y)}\frac{dy}{dx}$$, where k is constant,

we get $$\frac{dv}{dx} + kPv = kQ$$

which is linear differential equation.

§ 20.4. Solution by Inspection :

(i) $d(xy) = xdy + ydx$

(ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(iv) $d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$

(v) $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$

(vi) $d\left(\frac{x^2}{y^2}\right) = \frac{2xy^2dx - 2x^2y\,dy}{y^4}$

(vii) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2xy^2dx}{x^4}$

(viii) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$

(ix) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(x) $d[ln(xy)] = \frac{xdy + ydx}{xy}$

(xi) $d\left[ln\left(\frac{x}{y}\right)\right] = \frac{ydx - xdy}{xy}$

(xii) $d\left[\frac{1}{2}ln(x^2 + y^2)\right] = \frac{xdx + ydy}{x^2 + y^2}$

(xiii) $d\left[ln\left(\frac{y}{x}\right)\right] = \frac{xdy - y\,dx}{xy}$

(xiv) $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$

(xv) $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^xdy}{y^2}$

(xvi) $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy - e^ydx}{x^2}$

(xvii) $d(x^my^n) = x^{m-1}\,y^{n-1}\,(mydx + nxdy).$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If $y = f(x)$ passing through (1, 2) satisfies the differential equation $y(1 + xy)\,dx - x\,dy = 0$, then

(a) $f(x) = \frac{2x}{2 - x^2}$ (b) $f(x) = \frac{x + 1}{x^2 + 1}$

(c) $f(x) = \frac{x - 1}{4 - x^2}$ (d) $f(x) = \frac{4x}{1 - 2x^2}$

2. The differential equation representing the family of the curves $y^2 = 2c(x + \sqrt{c})$ where c is a positive parameter, is of

(a) order 1, degree 3

(b) order 2, degree 2

(c) order 3, degree 3

(d) order 4, degree 4

3. The solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi(y/x)}{\phi'(y/x)} \text{ is}$$

(a) $x\,\phi(y/x) = k$ (b) $\phi(y/x) = kx$

(c) $y\,\phi(y/x) = k$ (d) $\phi(y/x) = ky$

4. Let a and b be respectively the degree and order of the differential equation of the family of circles touching the lines $y^2 - x^2 = 0$ and lying in the first and second quadrant then

(a) $a = 1, b = 2$ (b) $a = 1, b = 1$

(c) $a = 2, b = 1$ (d) $a = 2, b = 2$

5. The integrating factor of the differential equation $\frac{dy}{dx}(x\log_e x) + y = 2\log_e x$ is given by

(a) x (b) e^x

(c) $\log_e x$ (d) $\log_e(\log_e x)$

6. A differential equation associated with the primitive $y = a + be^{5x} + ce^{-7x}$ is
(a) $y_3 + 2y_2 - y_1 = 0$
(b) $y_3 + 2y_2 - 35y_1 = 0$
(c) $4y_3 + 5y_2 - 20y_1 = 0$
(d) None of these

7. The function $f(t) = \frac{d}{dt}\int_0^t \frac{dx}{1 - \cos t \cos x}$ satisfies the differential equation
(a) $\frac{df}{dt} + 2f(t) \cot t = 0$
(b) $\frac{df}{dt} - 2f(t) \cot t = 0$
(c) $\frac{df}{dt} + 2f(t) = 0$
(d) $\frac{df}{dt} - 2f(t) = 0$

8. A continuously differentiable function $y = f(x)$, $x \in (0, \pi)$ satisfying $y' = 1 + y^2$, $y(0) = 0 = y(\pi)$ is
(a) $\tan x$
(b) $x(x - \pi)$
(c) $(x - \pi)(1 - e^x)$
(d) not possible

9. The largest value of c such that there exists a differential function $h(x)$ for $-c < x < c$ that is a solution of $y_1 = 1 + y^2$ with $h(0) = 0$ is
(a) 2π (b) π
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

10. The form of the differential equation of all central conics is
(a) $x = y\frac{dy}{dx}$
(b) $x + y\frac{dy}{dx} = 0$
(c) $x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = y\frac{dy}{dx}$
(d) None of these

11. The particular solution of the differential equation $y' + 3xy = x$ which passes through (0, 4) is
(a) $y = 1 - 11\, e^{-3x^2/2}$
(b) $3y = 1 + 11\, e^{-3x^2/2}$
(c) $3y = 1 - 11\, e^{-3x^2/2}$
(d) None of these

12. The particular solution of $l\,n\left(\frac{dy}{dx}\right) = 3x + 4y$, $y(0) = 0$, is
(a) $3e^{3x} + 4e^{-4y} = 7$
(b) $3e^{4y} - 4e^{-3x} = 7$
(c) $4e^{3x} + 3e^{-4y} = 7$
(d) None of these

13. If the slope of tangent to the curve is maximum at $x = 1$ and curve has a minimum value 1 at $x = 0$, then the curve which also satisfies the equation $y''' = 4x - 3$ is
(a) $y + 2x + \frac{dy}{dx} = 0$
(b) $y = 1 + \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{6}$
(c) $y = 1 + x + x^2 + x^3$
(d) None of these

14. A spherical raindrop evaporates at a rate proportional to its surface area at any instant t. The differential equation giving the rate of change of the radius of the raindrop is
(a) $\frac{d^2r}{dt^2} + 2r = 0$
(b) $\frac{d^2r}{dt^2} - 3r = 0$
(c) $\frac{d^2r}{dt^2} = 0$
(d) None of these

15. Through any point (x, y) of a curve which passes through the origin, lines are drawn parallel to the co-ordinate axes. The curve, given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is twice the other, represents a family of
(a) circle
(b) parabola
(c) ellipse
(d) hyperbola

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer (s). For each question write the letters a, b, c, d corresponding to the correct answer (s).

16. The degree of the differential equation satisfying $(\sqrt{1+x^2}+\sqrt{1+y^2}) = A(x\sqrt{1+y^2} - y\sqrt{1+x^2})$ is
(a) 2 (b) 3
(c) 4 (d) None of these

17. Solution of $2y\sin x \frac{dy}{dx} = 2\sin x\cos x - y^2\cos x$, $x = \frac{\pi}{2}$, $y = 1$ is given by
(a) $y^2 = \sin x$
(b) $y = \sin^2 x$
(c) $y^2 = 1 + \cos x$
(d) None of these

18. The differential equation of all 'Simple Harmonic Motions' of given period $\frac{2\pi}{n}$ is $\frac{2\pi}{\omega}$, $x = a\cos(\omega t + \phi)$
(a) $\frac{d^2x}{dt^2} + nx = 0$ (b) $\frac{d^2x}{dt^2} + n^2x = 0$
(c) $\frac{d^2x}{dt^2} - n^2x = 0$ (d) $\frac{d^2x}{dt^2} + \frac{1}{n^2}x = 0$

19. The slope of the tangent at (x, y) to a curve passing through $(1, \pi/4)$ is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$ then the equation of the curve is
(a) $y = \tan^{-1}\left(ln\left(\frac{e}{x}\right)\right)$
(b) $y = x\tan^{-1}\left(ln\left(\frac{x}{e}\right)\right)$
(c) $y = x\tan^{-1}\left(ln\left(\frac{e}{x}\right)\right)$
(d) None of these

20. The differential equation of the curve $\frac{x}{c-1} + \frac{y}{c+1} = 1$ is given by
(a) $(y'-1)(y+xy') = 2y'$
(b) $(y'+1)(y-xy') = y'$
(c) $(y'+1)(y-xy') = 2y'$
(d) None of these

21. Solution of the differential equation $x\cos x\left(\frac{dy}{dx}\right) + y(x\sin x + \cos x) = 1$ is
(a) $xy = \sin x + c\cos x$
(b) $xy\sec x = \tan x + c$
(c) $xy + \sin x + c\cos x = 0$
(d) None of these

22. Solution of the differential equation $(1+y^2)\cdot dx = (\tan^{-1}y - x)\,dy$ is
(a) $xe^{\tan^{-1}y} = (1-\tan^{-1}y)\,e^{\tan^{-1}y} + c$
(b) $xe^{\tan^{-1}y} = (\tan^{-1}y - 1)\,e^{\tan^{-1}y} + c$
(c) $x = \tan^{-1}y - 1 + c\,e^{-\tan^{-1}y}$
(d) None of these

23. Equation of the curve in which the subnormal is twice the square of the ordinate is given by
(a) $\log y = 2x + \log c$ (b) $y = ce^{2x}$
(c) $\log y = 2x - \log c$ (d) None of these

24. If $\phi(x) = \int \{\phi(x)\}^{-2}\,dx$ and $\phi(1) = 0$ then $\phi(x) =$
(a) $\{2(x-1)\}^{1/4}$
(b) $\{5(x-2)\}^{1/5}$
(c) $\{3(x-1)\}^{1/3}$
(d) None of these

25. Solution of differential equation $(2x\cos y + y^2\cos x)\,dx + (2y\sin x - x^2\sin y)\,dy = 0$ is
(a) $x^2\cos y + y^2\sin x = c$
(b) $x\cos y - y\sin x = c$
(c) $x^2\cos^2 y + y^2\sin^2 x = c$
(d) None of these

Practice Test

M.M. : 10 Time : 15 Min

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. Solution of the differential equation $\sin y \frac{dy}{dx} = \cos(1 - x\cos y)$ is

 (a) $\sec y = x - 1 - ce^x$

 (b) $\sec y = x + 1 + ce^x$

 (c) $y = x + e^x + c$

 (d) None of these

2. The differential equation whose solution is $(x - h)^2 + (y - k)^2 = a^2$ is (a is a constant)

 (a) $\{1 + (y')^2\}^3 = a^3 y''$ (b) $\{1 + (y')^2\}^3 = a^2 (y'')^2$

 (c) $\{1 + (y')^3\} = a^2 (y'')^2$ (d) None of these

3. Solution of the equation $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$ is

 (a) $2x = \sin y (1 + 2cx^2)$

 (b) $2x = \sin y (1 + cx^2)$

 (c) $2x + \sin y (1 + cx^2)$

 (d) None of these

4. Solution of the differential equation $\frac{dy}{dx} = \frac{x + y + 7}{2x + 2y + 3}$ is

 (a) $6(x + y) + 11 \log(3x + 3y + 10) = 9x + c$

 (b) $6(x + y) - 11 \log(3x + 3y + 10) = 9x + c$

 (c) $6(x + y) - 11 \log(x + y + 10/3) = 9x + c$

 (d) None of these

5. The solution of the equation $\frac{dy}{dx} - 3y = \sin 2x$ is

 (a) $y e^{-3x} = -\frac{1}{13} e^{-3x}(2\cos 2x + 3\sin 2x) + c$

 (b) $y = -\frac{1}{13}(2\cos 2x + 3\sin 2x) + ce^{3x}$

 (c) $y = \{-1/\sqrt{13}\}\cos(2x - \tan^{-1}(3/2)) + ce^{3x}$

 (d) $y = \{-1/\sqrt{13}\}\sin(2x + \tan^{-1}(2/3)) + ce^{3x}$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (b)
7. (a)	**8.** (a)	**9.** (c)	**10.** (c)	**11.** (b)	**12.** (c)
13. (b)	**14.** (c)	**15.** (b)			

Multiple Choice–II

16. (d)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (c)	**21.** (a), (b)
22. (b), (c)	**23.** (a), (b), (c)	**24.** (c)	**25.** (a)		

Practice Test

1. (b)	**2.** (b)	**3.** (a, b)	**4.** (b, c)	**5.** (a), (b), (c), (d)

CO-ORDINATE GEOMETRY

21

STRAIGHT LINE

§ 21.1. Distance Formula

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \left| \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \right|$$

Distance of (x_1, y_1) from origin $= \left| \sqrt{x_1^2 + y_1^2} \right|$

Note : If distance between two points is given then use ± sign.

§ 21.2. Section Formula

If $R(x, y)$ divides the join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m_1 : m_2$ $(m_1, m_2 > 0)$ then

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2};\ y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \qquad \text{(divides internally)}$$

and $$x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2};\ y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2} \qquad \text{(divides externally)}$$

§ 21.3. Area of a Triangle

The area of the triangle ABC with vertices $A(x_1, y_1)$; $B(x_2, y_2)$ and $C(x_3, y_3)$. The area of triangle ABC is denoted by Δ and is given as :

(i) $$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]$$

Note : (a) If one vertex (x_3, y_3) is at (0, 0), then

$$\Delta = \frac{1}{2} \left| (x_1 y_2 - x_2 y_1) \right|$$

(b) If area of triangle is given then use ± sign

(ii) If $a_r x + b_r y + c_r = 0$, $(r = 1, 2, 3)$ are the sides of a triangle, then the area of the triangle is given by

$$\Delta = \frac{1}{2C_1 C_2 C_3} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where C_1, C_2, C_3 are the co-factors of c_1, c_2, c_3 in the determinant.

§ 21.4. Area of Polygon

The area of the polygon whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots (x_n, y_n)$ is

$$A = \frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \dots + (x_n y_1 - x_1 y_n)]$$

§ 21.5. Nine point Centre

The centre of **nine point** circle (which passes through the feet of the perpendiculars, the middle points of the sides, and the middle points of the lines joining the angular points to the orthocentre) lies on *OP* and bisects it. When *O* and *P* are the orthocentre and circumcentre of a triangle. Also the radius of the **nine-point** circle of a triangle is half the radius of the circumcircle.

§ 21.6. Relation in Orthocentre, Nine point centre, Centroid and Circumcentre

The orthocentre, the nine point centre, the centroid and the circumcentre therefore all lie on a straight line and centroid divides the orthocentre and circumcentre in the ratio 2 : 1 (Internally).

§ 21.7. Collinearity of three Given Points

The three given points are collinear *i.e.*, lie on the same straight line if

(i) Area of triangle *ABC* is zero.

(ii) Slope of *AB* = slope of *BC* = slope of *AC*

(iii) Distance between *A* & *B* + distance between *B* & *C* = distance between *A* & *C*

(iv) Find the equation of the line passing through any two points, if third point satisfied the equation of the line then three points are collinear.

§ 21.8. Locus

The locus of a moving point is the path traced out by that point under one or more given conditions.

How to find the locus of a point : Let (x_1, y_1) be the co-ordinates of the moving points say *P*. Now apply the geometrical conditions on x_1, y_1. This gives a relation between x_1 and y_1. Now replace x_1 by x and y_1 by y in the eleminant and resulting equation would be the equation of the locus.

§ 21.9. Positions of Points (x_1, y_1) and (x_2, y_2) Relative to a given Line

If the points (x_1, y_1) and (x_2, y_2) are on the same side of the line $ax + by + c = 0$, then $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ both are of the same sign and hence $\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$, and if the points (x_1, y_1) & (x_2, y_2) are on the opposite sides of the line $ax + by + c = 0$, then $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ both are of signs opposite to each other and hence

$$\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

§ 21.10. Angle between two Lines

Angle between two lines whose slopes are m_1 and m_2 is $\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Corollary 1 : If two lines whose slopes are m_1 and m_2 are parallel iff $\theta = 0$ (or π) $\Leftrightarrow \tan\theta = 0 \Leftrightarrow m_1 = m_2$

Corollary 2 : If two lines whose slopes are m_1 and m_2 are perpendicular iff $\theta = \frac{\pi}{2}\left(\text{or} -\frac{\pi}{2}\right)$ $\Rightarrow \cot\theta = 0 \Leftrightarrow m_1 m_2 = -1$

Note : Two lines given by the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c = 0$ are

(i) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ (slopes are equal)

(ii) Perpendicular if $a_1a_2 + b_1b_2 = 0$ (Product of their slopes is -1)

(iii) Identical if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (compare with the conditions in (i))

§ 21.11. Equation of a Line Parallel to a given Line

The equation of a line parallel to the line $ax + by + c = 0$ is of the form $ax + by + k = 0$, where k is any number.

§ 21.12. Equation of a line Perpendicular to a given Line

The equation of a line perpendicular to the line $ax + by + c = 0$ is of the form $bx - ay + k = 0$ where k is any number.

§ 21.13. Length of Perpendicular from a Point on a Line

The length of perpendicular from (x_1, y_1) on $ax + by + c = 0$ is

$$\left| \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$$

§ 21.14. Distance between two Parallel Lines

Let the two parallel lines be $ax + by + c = 0$ and $ax + by + c_1 = 0$

First Method : The perpendicular distance between the lines is $\left| \frac{c_1 - c}{\sqrt{(a^2 + b^2)}} \right|$

Second Method : Find the co-ordinates of any point on one of the given lines, preferably putting $x = 0$ or $y = 0$. Then the perpendicular distance of this point from the other line is the required distance between the lines.

§ 21.15. A line Equally Inclined with two Lines

Let the two lines with slopes m_1 and m_2 be equally inclined to a line with slope m, then

$$\frac{m_1 - m}{1 + m_1 m} = -\left(\frac{m_2 - m}{1 + m_2 m}\right)$$

§ 21.16. Equations of Straight lines through (x_1, y_1) making $\angle \alpha$ with $y = mx + c$ are

$$y - y_1 = \tan(\theta - \alpha)(x - x_1)$$

and

$$y - y_1 = \tan(\theta + \alpha)(x - x_1)$$

where

$$\tan\theta = m.$$

§ 21.17. Family of Lines

Any line passing through the point of intersection of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ can be represented by the equation $(a_1 x + b_1 y + c_1) + \lambda (a_2 x + b_2 y + c_2) = 0$.

§ 21.18. Concurrent Lines

The three given lines are concurrent if they meet in a point. Hence to prove that three given lines are concurrent, we proceed as follows :

Method I : Find the point of intersection of any two lines by solving them simultaneously. If this point satisfies the third equation also, then the given lines are concurrent.

Method II : The three lines $a_i x + b_i y + c_i = 0$, $i = 1, 2, 3$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Method III : The condition for the lines $P = 0$, $Q = 0$, and $R = 0$ to be concurrent is that three constants l, m, n (not all zeros at the same time) can be obtained such that

$$lP + mQ + nR = 0.$$

§ 21.19. The Image of a point with Respect to the Line Mirror :

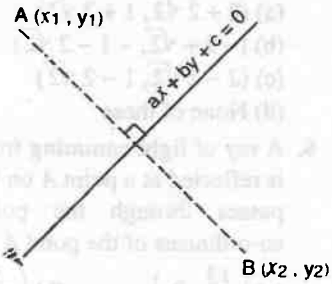


Fig.21.1

The image of $A(x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$ be $B(x_2, y_2)$ is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

21.20. Equations of the Bisectors of the Angles between two Lines

Equations of the bisectors of the lines

$L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$

$(a_1b_2 \neq a_2b_1)$ where $c_1 > 0$ & $c_2 > 0$ are

$$\frac{(a_1x + b_1y + c_1)}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{(a_2x + b_2y + c_2)}{\sqrt{(a_2^2 + b_2^2)}}$$

Conditions	Acute angle bisector	Obtuse angle bisector
$a_1a_2 + b_1b_2 > 0$	–	+
$a_1a_2 + b_1b_2 < 0$	+	–

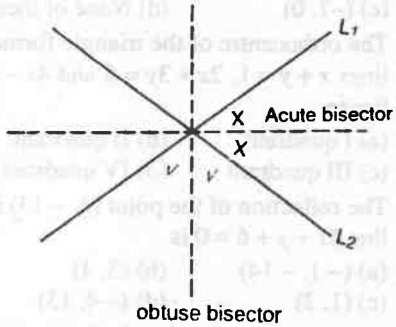


Fig.21.2

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If the distance of any point (x, y) from the origin is defined as
$$d(x, y) = \max\{|x|, |y|\},$$
$d(x, y) = a$ non zero constant, then the locus is
(a) a circle (b) a straight line
(c) a square (d) a triangle

2. The point (4, 1) undergoes the following three transformations successively
(I) Reflection about the line $y = x$.
(II) Transformation through a distance 2 units along the positive direction of x-axis
(III) Rotation through an angle $\pi/4$ about the origin in the anticlockwise direction.
The final position of the point is given by the co-ordinates
(a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $(-2, 7\sqrt{2})$
(c) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (d) $(\sqrt{2}, 7\sqrt{2})$

3. One of the bisector of the angle between the lines
$a(x-1)^2 + 2h(x-1)(y-1) + b(y-2)^2 = 0$
is $x + 2y - 5 = 0$. The other bisector is
(a) $2x - y = 0$ (b) $2x + y = 0$
(c) $2x + y - 4 = 0$ (d) $x - 2y + 3 = 0$

4. Line L has intercepts a and b on the co-ordinate axes, when the axes are rotated through a given angle; keeping the origin fixed, the same line has intercepts p and q, then
(a) $a^2 + b^2 = p^2 + q^2$ (b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$
(c) $a^2 + p^2 = b^2 + q^2$ (b) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$

5. The point $A(2, 1)$ is translated parallel to the line $x - y = 3$ by a distance 4 units. If the new position A' is in third quadrant, then the co-ordinates of A' are

(a) $(2+2\sqrt{2}, 1+2\sqrt{2})$
(b) $(-2+\sqrt{2}, -1-2\sqrt{2})$
(c) $(2-2\sqrt{2}, 1-2\sqrt{2})$
(d) None of these

6. A ray of light comming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). The co-ordinates of the point A is

(a) $\left(\frac{13}{5}, 0\right)$ (b) $\left(\frac{5}{13}, 0\right)$
(c) (–7, 0) (d) None of these

7. The orthocentre of the triangle formed by the lines $x+y=1$, $2x+3y=6$ and $4x-y+4=0$ lies in,

(a) I quadrant (b) II quadrant
(c) III quadrant (d) IV quadrant

8. The reflection of the point (4, – 13) in the the line $5x+y+6=0$ is

(a) (– 1, – 14) (b) (3, 4)
(c) (1, 2) (d) (– 4, 13)

9. All the points lying inside the triangle formed by the points (0, 4), (2, 5) and (6, 2) satisfy

(a) $3x+2y+8>0$
(b) $2x+y-10>0$
(c) $2x-3y-11>0$
(d) $-2x+y-3>0$

10. A line passing through $P(4, 2)$ meets the x and y-axis at A and B respectively. If O is the origin, then locus of the centre of the circumcircle of ΔOAB is

(a) $x^{-1}+y^{-1}=2$ (b) $2x^{-1}+y^{-1}=1$
(c) $x^{-1}+2y^{-1}=1$ (d) $2x^{-1}+2y^{-1}=1$

11. Let n be the number of points having rational co-ordinates equidistant from the point $(0, \sqrt{3})$ then

(a) $n\le 1$ (b) $n=1$
(c) $n<2$ (d) $n>2$

12. The co-ordinates of the middle points of the sides of a triangle are (4, 2), (3, 3) and (2, 2), then the co-ordinates of its centroid are

(a) (3, 7/3) (b) (3, 3)
(c) (4, 3) (d) None of these

13. The line segment joining the points (1, 2) and (–2, 1) is divided by the line $3x+4y=7$ in the ratio

(a) 3 : 4 (b) 4 : 3
(c) 9 : 4 (d) 4 : 9

14. If a straight line passes through (x_1, y_1) and its segment between the axes is bisected at this point, then its equation is given by

(a) $\frac{x}{x_1}+\frac{y}{y_1}=2$ (b) $2(xy_1+yx_1)=x_1y_1$
(c) $xy_1+yx_1=x_1y_1$ (d) None of these

15. If p_1, p_2, p_3 be the perpendiculars from the points $(m^2, 2m)$, $(mm', m+m')$ and $(m'^2, 2m')$ respectively on the line

$$x\cos\alpha+y\sin\alpha+\frac{\sin^2\alpha}{\cos\alpha}=0$$

then p_1, p_2, p_3 are in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

16. The acute angle θ through which the co-ordinate axes should be rotated for the point $A(2, 4)$ to attain the new abscissa 4 is given by

(a) $\tan\theta=3/4$ (b) $\tan\theta=5/6$
(c) $\tan\theta=7/8$ (d) None of these

17. If $\frac{2}{1!\,9!}+\frac{2}{3!\,7!}+\frac{1}{5!\,5!}=\frac{2^m}{n!}$ then orthocentre of the triangle having sides $x-y+1=0$, $x+y+3=0$ and $2x+5y-2=0$ is

(a) $(2m-2n, m-n)$ (b) $(2m-2n, n-m)$
(c) $(2m-n, m+n)$ (d) $(2m-n, m-n)$

18. The equation of straight line equally inclined to the axes and equidistant from the point (1, –2) and (3, 4) is

(a) $x+y=1$ (b) $y-x-1=0$
(c) $y-x=2$ (d) $y-x+1=0$

19. If one of the diagonal of a square is along the line $x=2y$ and one of its vertices is (3, 0) then its sides through this vertex are given by the equations

(a) $y-3x+9=0$, $3y+x-3=0$
(b) $y+3x+9=0$, $3y+x-3=0$
(c) $y-3x+9=0$, $3y-x+3=0$
(d) $y-3x+3=0$, $3y+x+9=0$

20. The graph of the function

$$y=\cos x\cos(x+2)-\cos^2(x+1) \text{ is}$$

(a) A straight line passing through $(0, -\sin^2 1)$ with slope 2
(b) A straight line passing through (0, 0)
(c) A parabola with vertex $(1, -\sin^2 1)$
(d) A straight line passing through the point $\left(\frac{\pi}{2}, \sin^2 1\right)$ are paralle to the x-axis

21. $P(m, n)$ (where m, n are natural numbers) is any point in the interior of the quadrilateral formed by the pair of lines $xy - 0$ and the two lines $2x + y - 2 = 0$ and $4x + 5y = 20$. The possible number of positions of the point P is
(a) Six (b) Five
(c) Four (d) Eleven

22. The image of the point $A(1, 2)$ by the line mirror $y = x$ is the point B and the image of B by the line mirror $y = 0$ is the point (α, β) then
(a) $\alpha = 1, \beta = -2$ (b) $\alpha = 0, \beta = 0$
(c) $\alpha = 2, \beta = -1$ (d) None of these

23. $ABCD$ is a square whose vertices A, B, C and D are (0, 0), (2, 0), (2, 2) and (0, 2) respectively. This square is rotated in the $X - Y$ plane with an angle of 30° in anticlockwise direction about an axis passing through the vertex A the equation of the diagonal BD of this rotated square is .... If E is the centre of the square, the equation of the circumcircle of the triangle ABE is
(a) $\sqrt{3}\,x + (1 - \sqrt{3})\,y = \sqrt{3}, x^2 + y^2 = 4$
(b) $(1 + \sqrt{3})\,x - (1 - \sqrt{2})\,y = 2, x^2 + y^2 = 9$
(c) $(2 - \sqrt{3})\,x + y = 2(\sqrt{3} - 1), x^2 + y^2 - x\sqrt{3} - y = 0$
(d) None of these

24. The straight line $y = x - 2$ rotates about a point where it cuts x-axis and becomes perpendicular on the straight line $ax + by + c = 0$ then its equation is
(a) $ax + by + 2a = 0$
(b) $ay - bx + 2b = 0$
(c) $ax + by + 2b = 0$
(d) None of these

25. The line $x + y = a$ meets the axis of x and y at A and B respectively a triangle AMN is inscribed in the triangle OAB, O being the origin, with right angle at N, M and N lie respectively on OB and AB. If the area of the triangle AMN is 3/8 of the area of the triangle OAB, then $AN/BN =$
(a) 1 (b) 2
(c) 3 (d) 4

26. If two vertices of an equilateral triangle have integral co-ordinates then the third vertex will have
(a) integral co-ordinates
(b) co-ordinates which are rational
(c) at least one co-ordinate irrational
(d) co-ordinates which are irrational

27. If $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are the values of n for which $\sum_{r=0}^{n-1} x^{2r}$ is divisible by $\sum_{r=0}^{n-1} x^r$, then the triangle having vertices (α_1, β_1), (α_2, β_2) and (α_3, β_3) can not be
(a) an isosceles triangle
(b) a right angled isosceles triangle
(c) a right angled triangle
(d) an equilateral triangle

28. The points $(\alpha, \beta), (\gamma, \delta), (\alpha, \delta)$ and (γ, β) where $\alpha, \beta, \gamma, \delta$ are different real numbers are
(a) collinear
(b) vertices of a square
(c) vertices of a rhombus
(d) concyclic

29. The equations of the three sides of a triangle are $x = 2$, $y + 1 = 0$ and $x + 2y = 4$. The co-ordinates of the circumcentre of the triangle are
(a) (4, 0) (b) (2, –1)
(c) (0, 4) (d) None of these

30. If $P(1 + \alpha/\sqrt{2}, 2 + \alpha/\sqrt{2})$ be any point on a line then the range of values of t for which the point P lies between the parallel lines $x + 2y = 1$ and $2x + 4y = 15$ is
(a) $-\frac{4\sqrt{2}}{3} < \alpha < \frac{5\sqrt{2}}{6}$
(b) $0 < \alpha < \frac{5/\sqrt{2}}{6}$
(c) $-\frac{4\sqrt{2}}{3} < \alpha < 0$
(d) None of these

31. If the point (a, a) fall between the lines $|x + y| = 2$ then
(a) $|a| = 2$ (b) $|a| = 1$

(c) $|a| < 1$ (d) $|a| < \frac{1}{2}$

32. If a ray travelling the line $x = 1$ gets reflected the line $x + y = 1$ then the equation of the line along which the reflected ray trevels is

(a) $y = 0$ (b) $x - y = 1$

(c) $x = 0$ (d) None of these

33. If $f(x + y) = f(x) f(y) \forall x, y \in R$ and $f(1) = 2$ then area enclosed by $3|x| + 2|y| \le 8$ is

(a) $f(4)$ sq. units

(b) $1/2 f(6)$ sq. units

(c) $\frac{1}{3} f(6)$ sq. units

(d) $\frac{1}{3} f(5)$ sq. units

34. The diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n' = 0$, $mx + ly + n = 0$, $mx + ly + n' = 0$ include an angle

(a) $\pi/3$ (b) $\pi/2$

(c) $\tan^{-1}\left(\frac{l^2 - m^2}{l^2 + m^2}\right)$ (d) $\tan^{-1}\left(\frac{2lm}{l^2 + m^2}\right)$

35. The area of the triangle having vertices (–2, 1), (2, 1) and $\left(\lim_{m \to \infty} \lim_{n \to \infty} \cos^{2m}(n!\,\pi x); x\right.$ is rational, $\lim_{m \to \infty} \lim_{n \to \infty} \cos^{2m}(n!\,\pi x)$; where x is irrational) is

(a) 2 (b) 3

(c) 4 (d) None of these

36. Consider the straight line $ax + by = c$ where $a, b, c \in R^+$ this line meets the co-ordinate axes at A and B respectively. If the area of the ΔOAB, O being origin, does not depend upon a, b and c then

(a) a, b, c are in A.P.

(b) a, b, c are in G.P.

(c) a, b, c are in H.P.

(d) None of these

37. If A and B are two points having co-ordinates (3, 4) and (5, –2) respectively and P is a point such that $PA = PB$ and area of triangle $PAB = 10$ square units, then the co-ordinates of P are

(a) (7, 4) or (13, 2) (b) (7, 2) or (1, 0)

(c) (2, 7) or (4, 13) (d) None of these

38. The position of a moving point in the xy-plane at time is given by $(u \cos \alpha.t, u \sin \alpha\, t - \frac{1}{2} gt^2)$, where u, α, g are constants. The locus of the moving point is

(a) a circle (b) a parabola

(c) an ellipse (d) None of these

39. If the lines $x + ay + a = 0$, $bx + y + b = 0$ and $cx + cy + 1 = 0$ (a, b, c being distinct $\neq 1$) are concurrent, then the value of $\frac{a}{a-1} + \frac{b}{b-1} + \frac{c}{c-1}$ is

(a) –1 (b) 0

(c) 1 (d) None of these

40. The medians AD and BE of the triangle with vertices $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are mutually perpendicular if

(a) $b = \sqrt{2}\, a$ (b) $a = \sqrt{2}\, b$

(c) $b = -\sqrt{2}\, a$ (d) $a = 5\sqrt{2}\, b$

41. P is a point on either of two lines $y - \sqrt{3}|x| = 2$ at a distance of 5 units from their point of intersection. The co-ordinates of the foot of perpendicular from P on the bisector of the angle between them are

(a) $\left(0, \frac{1}{2}(4 + 5\sqrt{3})\right)$ or $\left(0, \frac{1}{2}(4 - 5\sqrt{3})\right)$, (depending on which line the point P is taken)

(b) $\left(0, \frac{1}{2}(4 + 5\sqrt{3})\right)$

(c) $\left(0, \frac{1}{2}(4 - 5\sqrt{3})\right)$

(d) $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

42. Let AB be a line segment of length 4 with the point A on the line $y = 2x$ and B on the line $y = x$. Then locus of middle point of all such line segment is a

(a) parabola (b) ellipse

(c) hyperbola (c) circle

43. Let O be the origin, and let $A(1, 0)$, $B(0, 1)$ be two points. If $P(x, y)$ is a point such that $xy > 0$ and $x + y < 1$ then

(a) P lies either inside ΔOAB or in third quadrant

(b) P can not be inside ΔOAB

(c) P lies inside the ΔOAB

(d) None of these

44. If the point $P(a^2, a)$ lies in the region corresponding to the acute angle between the lines $2y = x$ and $4y = x$, then

(a) $a \in (2, 4)$ (b) $a \in (2, 6)$

(c) $a \in (4, 6)$ (d) $a \in (4, 8)$

45. A ray of light coming along the line $3x + 4y - 5 = 0$ gets reflected from the line $ax + by - 1 = 0$ and goes along the line $5x - 12y - 10 = 0$ then

(a) $a = \frac{64}{115}, b = \frac{112}{15}$

(b) $a = -\frac{64}{115}, b = \frac{8}{115}$

(c) $a = \frac{64}{115}, b = -\frac{8}{115}$

(d) $a = -\frac{64}{115}, b = -\frac{8}{115}$

46. If line $2x + 7y - 1 = 0$ intersect the lines $L_1 \equiv 3x + 4y + 1 = 0$ and $L_2 = 6x + 8y - 3 = 0$ in A and B respectively, then equation of a line parallel to L_1 and L_2 and passes through a point P such that $AP : PB = 2 : 1$ (internally) is (P is on the line $2x + 7y - 1 = 0$)

(a) $9x + 12y + 3 = 0$ (b) $9x + 12y - 3 = 0$

(c) $9x + 12y - 2 = 0$ (d) None of these

47. If the point $(1 + \cos\theta, \sin\theta)$ lies between the region corresponding to the acute angle between the lines $x - 3y = 0$ and $x - 6y = 0$ then

(a) $\theta \in R$

(b) $\theta \in R \sim n\pi, n \in I$

(c) $\theta \in R \sim \left(n\pi + \frac{\pi}{2}\right), n \in I$

(d) None of these

48. In a ΔABC, side AB has the equation $2x + 3y = 29$ and the side AC has the equation $x + 2y = 16$. If the mid point of BC is (5, 6) then the equation of BC is

(a) $2x + y = 7$ (b) $x + y = 1$

(c) $2x - y = 17$ (d) None of these

49. The locus of a point which moves such that the square of its distance from the base of an isosceles triangle is equal to the rectangle under its distances from the other two sides is

(a) a circle (b) a parabola

(c) an ellipse (d) a hyperbola

50. The equation of a line through the point (1, 2) whose distance from the point (3, 1) has the greatest possible value is

(a) $y = x$ (b) $y = 2x$

(c) $y = -2x$ (d) $y = -x$

51. If the point $(\cos\theta, \sin\theta)$ does not fall in that angle between the lines $y = |x - 1|$ in which the origin lies then θ belongs to

(a) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

(b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $(0, \pi)$

(d) None of these

52. ABC is an equilateral triangle such that the vertices B and C lie on two parallel lines at a distance 6. If A lies between the parallel lines at a distance 4 from one of them then the length of a side of the equilateral triangle is

(a) 8 (b) $\sqrt{\frac{88}{3}}$

(c) $\frac{4\sqrt{7}}{\sqrt{3}}$ (d) None of these

53. The four sides of a quadrilateral are given by the equation $xy(x - 2)(y - 3) = 0$. The equation of the line parallel to $x - 4y = 0$ that divides the quadrilateral in two equal areas is

(a) $x - 4y - 5 = 0$ (b) $x - 4y + 5 = 0$

(c) $x - 4y - 1 = 0$ (d) $x - 4y + 1 = 0$

54. Two points A and B move on the x-axis and the y-axis respectively such that the distance between the two points is always the same. The locus of the middle point of AB is

(a) a straight line (b) a circle

(c) a parabola (d) an ellipse

55. If t_1, t_2 and t_3 are distinct, the points $(t_1, 2at_1 + at_1^3), (t_2, 2at_2 + at_2^3)$ and $(t_3, 2at_3 + at_3^3)$ are collinear if

(a) $t_1 t_2 t_3 = -1$ (b) $t_1 + t_2 + t_3 = t_1 t_2 t_3$

(c) $t_1 + t_2 + t_3 = 0$ (d) $t_1 + t_2 + t_3 = -1$

56. If sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is

(a) a square (b) a circle

(c) a straight line (d) two intersecting lines

57. If the lines $2(\sin a+\sin b)x-2\sin(a-b)y=3$ and $2(\cos a+\cos b)x+2\cos(a-b)y=5$ are perpendicular, then $\sin 2a+\sin 2b$ is equal to
(a) $\sin(a-b)-2\sin(a+b)$
(b) $\sin(2a-2b)-2\sin(a+b)$
(c) $2\sin(a-b)-\sin(a+b)$
(d) $\sin(2a-2b)-\sin(a+b)$

58. The line $x+y=1$ meets x-axis at A and y-axis at B, P is the mid point of AB fig. No. 23.3 P_1 is the foot of the perpendicular from P to OA; M_1 is that of P_1 from OP; P_2 is that of M_1 from OA; M_2 is that of P_2 from OP; P_3 is that of M_2 from OA and so on. If P_n denotes the nth foot of the perpendicular on OA from M_{n-1}, then $OP_n=$
(a) $1/2n$ (b) $1/2^n$
(c) 2^n-1 (d) 2^n+3

59. If the area of the triangle whose vertices are (b, c), (c, a) and (a, b) is Δ, then the area of triangle whose vertices are $(ac-b^2, ab-c^2)$, $(ba-c^2, bc-a^2)$ and $(cb-a^2, ca-b^2)$ is
(a) Δ^2 (b) $(a+b+c)^2\Delta$
(c) $a\Delta+b\Delta^2$ (d) None of these

60. On the portion of the straight line $x+y=2$ which is intercepted between the axes, a square is constructed away from the origin, with this portion as one of its side. If p denote the perpendicular distance of a side of this square from the origin, then the maximum value of p is
(a) $\sqrt{2}$ (b) $2\sqrt{2}$
(c) $3\sqrt{2}$ (d) $4\sqrt{2}$

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

61. The points (2, 3), (0, 2), (4, 5) and (0, t) are concyclic if the value of t is
(a) 1 (b) 1
(c) 17 (d) 3

62. The point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ lies on
(a) $x-y=0$
(b) $(x+y)(a+b)=2ab$
(c) $(lx+my)(a+b)=(l+m)ab$
(d) $(lx-my)(a+b)=(l-m)ab$

63. The equations
$$(b-c)x+(c-a)y+a-b=0$$
$$(b^3-c^3)x+(c^3-a^3)y+a^3-b^3=0$$
will represent the same line if
(a) $b=c$ (b) $c=a$
(c) $a=b$ (d) $a+b+c=0$

64. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y=x+3$. The co-ordinates of the third vertex can be
(a) $(-3/2, 3/2)$ (b) $(3/4, -3/2)$
(c) $(7/2, 13/2)$ (d) $(-1/4, 11/4)$

65. If (–6, –4), (3, 5), (–2, 1) are the vertices of a parallelogram then remaining vertex can not be
(a) (0, –1) (b) (–1, 0)
(c) (–11, –8) (d) (7, 10)

66. If the point $P(x, y)$ be equidistant from the points $A(a+b, a-b)$ and $B(a-b, a+b)$ then
(a) $ax=by$
(b) $bx=ay$
(c) $x^2-y^2=2(ax+by)$
(d) P can be (a, b)

67. If the lines $x-2y-6=0$, $3x+y-4=0$ and $\lambda x+4y+\lambda^2=0$ are concurrent, then
(a) $\lambda=2$ (b) $\lambda=-3$
(c) $\lambda=4$ (d) $\lambda=-4$.

68. Equation of a straight line passing through the point of intersection of $x-y+1=0$ and $3x+y-5=0$ are perpendicular to one of them is
(a) $x+y+3=0$ (b) $x+y-3=0$
(c) $x-3y-5=0$ (d) $x-3y+5=0$

69. If one vertex of an equilateral triangle of side a lies at the origin and the other lies on the

line $x - \sqrt{3}y = 0$, the co-ordinates of the third vertex are

(a) $(0, a)$ (b) $(\sqrt{3}\,a/2, -a/2)$

(c) $(0, -a)$ (d) $(-\sqrt{3}\,a/2, a/2)$

70. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent $(a + b + c \neq 0)$ then

(a) $a^3 + b^3 + c^3 - 3abc = 0$

(b) $a = b$

(c) $a = b = c$

(d) $a^2 + b^2 + c^2 - bc - ca - ab = 0$

71. If the co-ordinates of the vertices of a triangle are rational numbers then which of the following points of the triangle will always have rational co-ordinates

(a) centroid (b) incentre

(c) circumcentre (d) orthocentre

72. Let $S_1, S_2, \ldots$ be squares such that four each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm ?

(a) 7 (b) 8

(c) 9 (d) 10

73. A line passing through the point (2, 2) and the axes enclose an area λ. The intercepts on the axes made by the line are given by the two roots of

(a) $x^2 - 2|\lambda|x + |\lambda| = 0$

(b) $x^2 + |\lambda|x + 2|\lambda| = 0$

(c) $x^2 - |\lambda|x + 2|\lambda| = 0$

(d) None of these

74. If $bx + cy = a$, where a, b, c are the same sign, be a line such that the area enclosed by the line and the axes of reference is $\frac{1}{8}$ unit^2 then

(a) b, a, c are in G.P.

(b) $b, 2a, c$ are in G.P.

(c) $b, \frac{a}{2}, c$ are in A.P.

(d) $b, -2a, c$ are in G.P.

75. Consider the straight lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$. The line $6x + 6y + 7 = 0$ is

(a) bisector of the angle including origin

(b) bisector of acute angle

(c) bisector of obtuse angle

(d) None of these

76. Two roads are represented by the equations $y - x = 6$ and $x + y = 8$. An inspection bunglow has to be so constructed that it is at a distance of 100 from each of the roads. Possible location of the bunglow is given by

(a) $(100\sqrt{2} + 1, 7)$ (b) $(1 - 100\sqrt{2}, 7)$

(c) $(1, 7 + 100\sqrt{2})$ (d) $(1, 7 - 100\sqrt{2})$

77. Angles made with the x-axis by two lines drawn through the point (1, 2) cutting the line $x + y = 4$ at a distance $\sqrt{6}/3$ from the point (1, 2) are

(a) $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ (b) $-\frac{7\pi}{12}$ and $-\frac{11\pi}{12}$

(c) $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ (d) None of these

78. A line of fixed length $(a + b)$ moves so that its ends are always on two fixed perpendicular lines. The locus of the point which divides this line into portions of lengths a and b is

(a) a circle (b) an ellipse

(c) a hyperbola (d) None of these

79. If each of the points $(x_1, 4), (-2, y_1)$ lines on the line joining the points $(2, -1), (5, -3)$ then the point $P(x_1, y_1)$ lies on the line

(a) $6(x + y) - 25 = 0$

(b) $2x + 6y + 1 = 0$

(c) $2x + 3y - 6 = 0$

(d) $6(x + y) - 23 = 0$

80. Two vertices of a triangle are $(3, -2)$ and $(-2, 3)$ and its orthocentre is $(-6, 1)$. Then its third vetex is

(a) (1, 6) (b) (−1, 6)

(c) (1, −6) (d) None of these

81. Length of the median from B on AC where $A(-1, 3)$, $B(1, -1)$, $C(5, 1)$ is

(a) $\sqrt{18}$ (b) $\sqrt{10}$

(c) $2\sqrt{3}$ (d) 4

82. The point $P(1, 1)$ is translated parallel to $2x = y$ in the first quadrant through a unit distance. The co-ordinates of the new position of P are

(a) $\left(1 \pm \frac{2}{\sqrt{5}}, 1 \pm \frac{1}{\sqrt{5}}\right)$

(b) $\left(1 \pm \frac{1}{\sqrt{5}}, 1 \pm \frac{2}{\sqrt{5}}\right)$

(c) $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ (d) $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

83. The mid points of the sides of a triangle are (5, 0), (5, 12) and (0, 12). The orthocentre of this triangle is

(a) (0, 0) (b) (10, 0)

(c) (0, 24) (d) $\left(\frac{13}{3}, 8\right)$

84. The algebraic sum of the perpendicular distances from $A(a_1, b_1)$; $B(a_2, b_2)$ and $C(a_3, b_3)$ to a variable line is zero, then the line passes through

(a) the orthocentre of ΔABC

(b) the centroid of ΔABC

(c) the circumcentre of ΔABC

(d) None of these

85. One vertex of the equilateral triangle with centroid at the origin and one side as $x + y - 2 = 0$ is

(a) (− 1, − 1) (b) (2, 2)

(c) (− 2, − 2) (d) none of these

86. The point (4, 1) undergoes the following two successive transformations :

(a) reflection about the line $y = x$

(b) rotation through a distance 2 units along the positive x-axis.

Then the final co-ordinates of the point are

(a) (4, 3) (b) (3, 4)

(c) 1, 4) (d) (7/2 , 7/2)

87. The incentre of the triangle formed by the lines $x = 0$, $y = 0$ and $3x + 4y = 12$ is at

(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) (1, 1)

(c) $\left(1, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, 1\right)$

88. If (a, b) be an end of a diagonal of a square and the other diagonal has the equation $x - y = a$ then another vertex of the square can be

(a) $(a - b, a)$ (b) $(a, 0)$

(c) $(0, -a)$ (d) $(a + b, b)$

89. The points $(p + 1, 1)$, $(2p + 1, 3)$ and $(2p + 2, 2p)$ are colliner if

(a) $p = -1$ (b) $p = 1/2$

(c) $p = 2$ (d) $p = -\frac{1}{2}$

90. If $3a + 2b + 6c = 0$, the family of straight lines $ax + by + c = 0$ passes through a fixed point whose co-ordinates are given by

(a) (1/2, 1/3) (b) (2, 3)

(c) (3, 2) (d) (1/3, 1/2)

Practice Test

M.M.: 20 Time : 30 min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. Consider the equation $y - y_1 = m(x - x_1)$. If m and x_1 are fixed and different lines are drawn for different values of y_1, then

(a) The lines will pass through a fixed point

(b) There will be a set of parallel lines

(c) All the lines intersect the line $x = x_1$

(d) All the lines will be parallel to the line $y = x_1$

2. If the line $y = \sqrt{3}x$ cut the curve $x^3 + y^3 + 3xy + 5x^2 + 3y^2 + 4x + 5y - 1 = 0$ at the points A, B, C then $OA.OB.OC$ is

(a) $\frac{4}{13}(3\sqrt{3} - 1)$ (b) $3\sqrt{3} + 1$

(c) $\frac{2}{\sqrt{3}} + 7$ (d) None of these

3. Let $L_1 \equiv ax + by + a^3\sqrt{b} = 0$ and $L_2 \equiv bx - ay + b^3\sqrt{a} = 0$ be two straight lines. The equations of the bisectors of the angle formed by the loci. Whose equations are $\lambda_1 L_1 - \lambda_2 L_2 = 0$ and $\lambda_1 L_1 + \lambda_2 L_2 = 0$, λ_1 and λ_2 being non zero real numbers, are given by

(a) $L_1 = 0$ (b) $L_2 = 0$

(c) $\lambda_1 L_1 + \lambda_2 L_2 = 0$ (d) $\lambda_2 L_1 - \lambda_1 L_2 = 0$

4. If $P(1, 0)$, $Q(-1, 0)$ and $R(2, 0)$ are three given points then the locus of point S satisfying the relation

$$(SQ)^2 + (SR)^2 = 2(SP)^2 \text{ is}$$

(a) a straight line parallel to x-axis

(b) circle through origin

(c) circle with centre at the origin

(d) a straight line parallel to y-axis

5. The base BC of a triangle ABC is bisected at the point (p, q) and the equations to the sides AB and AC are $px + qy = 1$ and $qx + py = 1$. The equation of the median through A is

(a) $(p - 2q)x + (q - 2p)y + 1 = 0$

(b) $(p + q)(x + y) - 2 = 0$

(c) $(2pq - 1)(px + qy - 1)$
$= (p^2 + q^2 - 1)(qx + py - 1)$

(d) None of these

6. The set of values of 'b' for which the origin and the point $(1, 1)$ lie on the same side of the straight line $a^2x + aby + 1 = 0\ \forall\, a \in R$, $b > 0$ are

(a) $b \in (2, 4)$ (b) $b \in (0, 2)$

(c) $b \in [0, 2]$ (d) None of these

7. A family of lines is given by the equation $(3x + 4y + 6) + \lambda(x + y + 2) = 0$. The line situated at the greatest distance from the point $(2, 3)$ belonging to this family has the equation

(a) $15x + 8y + 30 = 0$

(b) $4x + 3y + 8 = 0$

(c) $5x + 3y + 6 = 0$

(d) $5x + 3y + 10 = 0$

8. The co-ordinates of the point P on the line $2x + 3y + 1 = 0$, such that $|PA - PB|$ is maximum where A is $(2, 0)$ and B is $(0, 2)$ is

(a) $(5, -3)$ (b) $(7, -5)$

(c) $(9, -7)$ (d) $(11, -9)$

9. The equation of the bisectors of the angles between the two intersecting lines:

$\dfrac{x-3}{\cos\theta} = \dfrac{y+5}{\sin\theta}$ and $\dfrac{x-3}{\cos\phi} = \dfrac{y+5}{\sin\phi}$ are

$\dfrac{x-3}{\cos\alpha} = \dfrac{y+5}{\sin\alpha}$ and $\dfrac{x-3}{\beta} = \dfrac{y+5}{\gamma}$ then

(a) $\alpha = \dfrac{\theta + \phi}{2}$ (b) $\beta = -\sin\alpha$

(c) $\gamma = \cos\alpha$ (d) $\beta = \sin\alpha$

10. If $4a^2 + 9b^2 - c^2 + 12ab = 0$, then the family of straight lines $ax + by + c = 0$ is either concurrent at ....or at ....

(a) $(2, 3)$ (b) $(-2, -3)$

(c) $(-3, -4)$ (d) $(3, -4)$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice–I

1. (b)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (a)
7. (a)	**8.** (a)	**9.** (a)	**10.** (b)	**11.** (c)	**12.** (a)
13. (d)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (d)
19. (a)	**20.** (c)	**21.** (c)	**22.** (c)	**23.** (c)	**24.** (b)
25. (c)	**26.** (c)	**27.** (d)	**28.** (b)	**29.** (a)	**30.** (a)
31. (c)	**32.** (a)	**33.** (c)	**34.** (b)	**35.** (a)	**36.** (b)
37. (b)	**38.** (b)	**39.** (c)	**40.** (a)	**41.** (b)	**42.** (b)

43. (a)	**44.** (a)	**45.** (c)	**46.** (c)	**47.** (d)	**48.** (b)
49. (d)	**50.** (b)	**51.** (b)	**52.** (c)	**53.** (b)	**54.** (b)
55. (c)	**56.** (c)	**57.** (b)	**58.** (b)	**59.** (b)	**60.** (c)

Multiple Choice–II

61. (a), (c)	**62.** (a), (b), (c), (d)		**63.** (a), (b), (c), (d)		**64.** (a), (c)
65. (a)	**66.** (b), (d)	**67.** (a), (d)	**68.** (b), (d)	**69.** (a), (b), (c), (d)	
70. (a), (b), (c), (d)		**71.** (a), (c), (d)	**72.** (b), (c), (d)	**73.** (c)	**74.** (b), (d)
75. (a), (b)	**76.** (a), (b), (c), (d)		**77.** (a), (b)	**78.** (a)	**79.** (b)
80. (b)	**81.** (b)	**82.** (b)	**83.** (a)	**84.** (b)	**85.** (c)
86. (b)	**87.** (b)	**88.** (b), (d)	**88.** (c), (d)	**90.** (a).	

Practice Test

1. (b), (c)	**2.** (a)	**3.** (a), (b)	**4.** (d)	**5.** (c)	**6.** (b)
7. (c)	**8.** (b)	**9.** (a), (b), (c)	**10.** (a), (b).		

22

PAIR OF STRAIGHT LINES

§ 22.1. Homogeneous Equation of Second Degree

An equation of the form $ax^2 + 2hxy + by^2 = 0$ is called a homogeneous equation of second degree. It represent two straight lines through the origin.

(i) The lines are real and distinct if $h^2 - ab > 0$.

(ii) The lines are coincident if $h^2 - ab = 0$

(iii) The lines are imaginary if $h^2 - ab < 0$

(iv) If the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$

then $$(y - m_1x)(y - m_2x) = y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = 0$$

$$\therefore \quad m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}.$$

§ 22.2. Angle Between two Lines

If $a + b \neq 0$ and θ is the acute angle between the lines whose joint equation is $ax^2 + 2hxy + by^2 = 0$, then

$$\tan\theta = \left|\frac{2\sqrt{(h^2 - ab)}}{a + b}\right|$$

§ 22.3. Equation of the Bisectors of the Angles Between the Lines $ax^2 + 2hxy + by^2 = 0$

The equation of bisectors is $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \qquad \text{...(i)}$$

Corollary 1 : If $a = b$, then the bisectors are $x^2 - y^2 = 0$, *i.e.*, $x - y = 0$, $x + y = 0$.

Corollary 2 : If $h = 0$, the bisectors are $xy = 0$, *i.e.*, $x = 0$, $y = 0$.

Corollary 3 : If in (i), coefficient of x^2 + coefficient of $y^2 = 0$, then the two bisectors are always perpendicular to each other.

§ 22.4. General equation of Second Degree

The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

is the general second degree equation and represents a conics (Pair of lines, circle, parabola, ellipse, hyperbola).

$\Rightarrow$ represents a pair of straight lines if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e., if $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

otherwise it represents a conic (*i.e.*, if $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$).

Corollary 1 : Angle between the lines : If the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, the angle θ between the lines is given by

$$\theta = \tan^{-1}\left|\frac{2\sqrt{(h^2 - ab)}}{(a+b)}\right|$$

these lines are parallel iff $h^2 = ab$ and perpendicular iff $a + b = 0$

Corollary 2 : Condition for coincidence of lines :

The lines will be coincident if $h^2 - ab = 0$, $g^2 - ac = 0$ and $f^2 - bc = 0$.

Corollary 3 : Point of intersection of the lines : The point of intersection of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$$

or
$$\left(\sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\frac{g^2 - ca}{h^2 - ab}\right)}\right)$$

22.5. Equation of the lines joining the origin to the points of intersection of a given line and a given curve :

Curve *PAQ* be (Fig. 22.1)

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

and the equation of the line *PQ* be

$$lx + my + n = 0 \quad \text{...(ii)}$$

From the equation of the line (ii) find the value of (1) in terms of x and y, *i.e.*

$$\frac{lx + my}{-n} = 1 \quad \text{...(iii)}$$

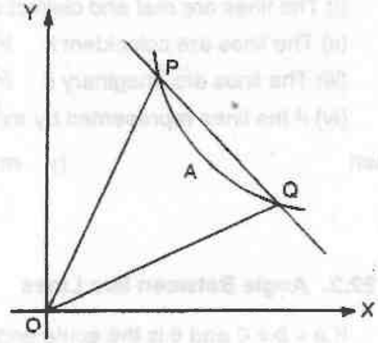


Fig. 22.1

Now the equation (i) can be written as

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)(1) + c(1)^2 = 0$$

$$\Rightarrow \quad ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0 \quad \text{...(iv)}$$

[replacing 1 by $\frac{lx + my}{-n}$ from (iii)]

Here the equation (iv) is homogeneous equation of second degree.

Above equation (iv) on simplification will be of the form $Ax^2 + 2Hxy + By^2 = 0$ and will represent the required straight lines.

If θ be the angle between them, then

$$\tan\theta = \left|\frac{2\sqrt{H^2 - AB}}{A + B}\right|$$

Hence the equation of pair of straight lines passing through the origin and the points of intersection of a curve and a line is obtained be making the curve homogeneous with the help of the line.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. Which of the following pair of straight lines intersect at right angles ?
(a) $2x^2 = y(x + 2y)$ (b) $(x + y)^2 = x(y + 3x)$
(c) $2y(x + y) = xy$ (d) $y = \pm 2y$

2. If co-ordinate axes are the angle bisectors of the pair of lines $ax^2 + 2hxy + by^2 = 0$, then
(a) $a = b$ (b) $h = 0$
(c) $a^2 + b = 0$ (d) $a + b^2 = 0$

3. If the two pairs of lines $x^2 - 2mxy - y^2 = 0$ and $x^2 - 2nxy - y^2 = 0$ are such that one of them represents the bisectors of the angles between the other, then
(a) $mn + 1 = 0$ (b) $mn - 1 = 0$
(c) $1/m + 1/n = 0$ (d) $1/m - 1/n = 0$

4. If the lines joining the origin to the points of intersection of $y = mx + 1$ with $x^2 + y^2 = 1$ are perpendicular, then
(a) $m = 1$ only (b) $m = \pm 1$
(c) $m = 0$ (d) None of these

5. If One of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between positive directions of the axes, a, b, h satisfy the relation
(a) $a + b = 2|h|$ (b) $a + b = -2h$
(c) $a - b = 2|h|$ (d) $(a - b)^2 = 4h^2$

6. The pair of lines joining the origin to the points of infersection of the curves
$ax^2 + 2hxy + by^2 + 2gx = 0$ and
$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$
will be at right angles to one another if
(a) $g(a' + b') = g'(a + b)$
(b) $g(a + b) = g'(a' + b')$
(c) $gg' = (a + b)(a' + b')$
(d) None of these

7. The equation $x^2 + 2\sqrt{2}\,xy + 2y^2 + 4x + 4\sqrt{2}\,y + 1 = 0$ represents a pair of lines. The distance between them
(a) 4 (b) $\frac{4}{\sqrt{3}}$
(c) 2 (d) $2\sqrt{3}$

8. The gradient of one of the lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, then :
(a) $h^2 = ab$ (b) $h - a + b$
(c) $8h^2 = 9ab$ (d) $ah^2 = 4ab$

9. The difference of the tangents of the angles which the lines $x^2(\sec^2\theta - \sin^2 0) - 2xy \tan\theta + y^2 \sin^2\theta = 0$ make with the x-axis is
(a) $2\tan\theta$ (b) 2
(c) $2\cot\theta$ (d) $\sin 2\theta$

10. If $\alpha, \beta > 0$ and $\alpha < \beta$ and $\alpha x^2 + 4\gamma xy + \beta y^2 + 4p(x + y + 1) = 0$ represents a pair of straight lines, then
(a) $\alpha < p < \beta$ (b) $p \le \alpha$
(c) $p \ge \alpha$ (d) $p < \alpha$ or $p > \beta$

11. The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $y = 0$ is
(a) $ax^2 - 2hxy - by^2 = 0$
(b) $bx^2 - 2hxy + ay^2 = 0$
(c) $bx^2 + 2hxy + ay^2 = 0$
(d) $ax^2 - 2hxy + by^2 = 0$

12. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines then the distance between them is
(a) $2\sqrt{\frac{g^2 - ac}{h^2 + a^2}}$ (b) $2\sqrt{\frac{g^2 - ac}{h^2 + a^2}}$
(c) $3\sqrt{\frac{g^2 + ac}{a(a + b)}}$ (d) $2\sqrt{\frac{g^2 + ac}{a(a + b)}}$

13. If the lines represented by $x^2 - 2pxy - y^2 = 0$ are rotated about the origin through an angle θ, one clockwise direction and other in anticlockwise direction, then the equation of the bisectors of the angle between the lines in the new position is
(a) $px^2 + 2xy - py^2 = 0$
(b) $px^2 + 2xy + py^2 = 0$

(c) $x^2 - 2pxy + y^2 = 0$

(d) None of these

14. If the sum of the slopes of the lines given by $4x^2 + 2\lambda xy - 7y^2 = 0$ is equal to the product of the slopes, then $\lambda =$

(a) –4 (b) 4

(c) –2 (d) 2

15. The pair of straight lines joining the origin to the common points of $x^2 + y^2 = 4$ and $y = 3x + c$ are perpendicular if $c^2 =$

(a) 20 (b) 13

(c) 1/5 (d) 5

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer (s). For each question, write the letters a, b, c, d corresponding to the correct answer (s).

16. If $x^2 + \alpha y^2 + 2\beta y = a^2$ represents a pair of perpendicular straight lines then :

(a) $\alpha = 1, \beta = a$

(b) $\alpha = 1, \beta = -a$

(c) $\alpha = -1, \beta = -a$

(d) $\alpha = -1, \beta = a$

17. Type of quadrilateral formed by the two pairs of lines $6x^2 - 5xy - 6y^2 = 0$ and $6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$ is

(a) square (b) rhombus

(c) parallelogram (d) rectangle

18. Two of the straight lines given by $3x^3 + 3x^2y - 3xy^2 + dy^3 = 0$ are at right angles if

(a) $d = -1/3$ (b) $d = 1/3$

(c) $d = -3$ (d) $d = 3$

19. If the line $y = mx$ is one of the bisector of the lines $x^2 + 4xy - y^2 = 0$, then the value of $m =$

(a) $\frac{-1+\sqrt{5}}{2}$ (b) $\frac{1+\sqrt{5}}{2}$

(c) $\frac{-1-\sqrt{5}}{2}$ (d) $\frac{1-\sqrt{5}}{2}$

20. If the angle between the two lines represented by

$$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$$

is $\tan^{-1}(m)$, then m is equal to

(a) 1/5 (b) –1

(c) –2/3 (d) None of these

21. If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is rotated about the origin through 90°, then their equations in the new position are given by

(a) $ax^2 - 2hxy + by^2 = 0$

(b) $ax^2 - 2hxy - by^2 = 0$

(c) $bx^2 - 2hxy + ay^2 = 0$

(d) $bx^2 + 2hxy + ay^2 = 0$

22. Products of the perpendiculars from (α, β) to the lines

$$ax^2 + 2hxy + by^2 = 0 \text{ is}$$

(a) $\frac{a\alpha^2 - 2h\,\alpha\beta + b\beta^2}{\sqrt{4h^2 + (a+b)^2}}$

(b) $\frac{a\,\alpha^2 - 2h\,\alpha\beta + b\beta^2}{\sqrt{4h^2 - (a-b)^2}}$

(c) $\frac{a\alpha^2 - 2h\alpha\beta + b\beta^2}{\sqrt{4h^2 - (a+b)^2}}$

(d) None of these

23. Equation of pair of straight lines drawn through (1, 1) and perpendicular to the pair of lines $3x^2 - 7xy - 2y^2 = 0$ is

(a) $2x^2 + 7xy - 11x + 6 = 0$

(b) $2(x-1)^2 + 7(x-1)(y-1) - 3y^2 = 0$

(c) $2(x-1)^2 + 7(x-1)(y-1) + 3(y-1)^2 = 0$

(d) None of these

24. Two pairs of straight lines have the equations $y^2 + xy - 12x^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$. One line will be common among them if

(a) $a = -3(2h + 3b)$ (b) $a = 8(h - 2b)$

(c) $a = 2(b + h)$ (d) $a = -3(b + h)$

25. The combined equation of three sides of a triangle is $(x^2 - y^2)(2x + 3y - 6) = 0$. If $(-2, a)$ is an interior point and $(b, 1)$ is an exterior point of the triangle then

(a) $2 < a < \frac{10}{3}$ (b) $-2 < a < \frac{10}{3}$

(c) $-1 < b < \frac{9}{2}$ (d) $-1 < b < 1$

Practice Test

M.M. : 10 Time : 15 Min

(A) There are 5 parts in this question. Each part has one or more than one correct answer(s). [5 × 2 = 10]

1. The equation of image of pair of lines $y = |x - 1|$ in y axis is

(a) $x^2 + y^2 + 2x + 1 = 0$

(b) $x^2 - y^2 + 2x - 1 = 0$

(c) $x^2 - y^2 + 2x + 1 = 0$

(d) none of these

2. Mixed term xy is to be removed from the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, one should rotate the axes through an angle θ given by $\tan 2\theta$ equal to

(a) $\frac{a-b}{2h}$ (b) $\frac{2h}{a+b}$

(c) $\frac{a+b}{2h}$ (d) $\frac{2h}{a-b}$

3. The equation $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$ represents

(a) circle

(b) pair of lines

(c) a parabola

(d) line segment $y = 0, -2 < x < 2$

4. If the two lines represented by $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ make angles α, β with the x-axis, then

(a) $\tan\alpha + \tan\beta = 4 \operatorname{cosec} 2\theta$

(b) $\tan\alpha \tan\beta = \sec^2\theta + \tan^2\theta$

(c) $\tan\alpha - \tan\beta = 2$

(d) $\frac{\tan\alpha}{\tan\beta} = \frac{2 + \sin 2\theta}{2 - \sin 2\theta}$

5. The equation $ax^2 + by^2 + cx + cy = 0$ represents a pair of straight lines if

(a) $a + b = 0$ (b) $c = 0$

(c) $a + c = 0$ (d) $c(a + b) = 0$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice–I

1. (a) **2.** (b) **3.** (a) **4.** (b) **5.** (b) **6.** (a)
7. (c) **8.** (c) **9.** (b) **10.** (d) **11.** (d) **12.** (b)
13. (a) **14.** (c) **15.** (a)

Multiple Choice–II

16. (c), (d) **17.** (a), (b), (c), (d) **18.** (d) **19.** (a), (c) **20.** (a)
21. (c) **22.** (d) **23.** (d) **24.** (a), (b) **25.** (a), (d)

Practice Test

1. (c) **2.** (d) **3.** (d) **4.** (a), (c), (d) **5.** (a), (b), (d)

23

CIRCLE

23.1. Definition

Circle is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is always is **constant.** The fixed point is called **the centre** and the constant distance is called the **radius** of the circle.

Some equations regarding circles :

(1) The equation of a circle with centre (h, k) and radius r is $(x-h)^2+(y-k)^2 = r^2$.
In particular, if the centre is at the origin, the equation, of circle is $x^2+y^2 = r^2$.
(2) Equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0.$$

(3) The general equation of a circle is

$$x^2+y^2+2gx+2fy+c = 0$$

where g, f, c are constants. The centre is $(-g, -f)$ and the radius is $\sqrt{(g^2+f^2-c)}$ $(g^2+f^2 \geq c)$.

Note : A general equation of second degree

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

in x, y represents a circle if

(i) Coefficient of x^2 = coefficient of y^2 *i.e.*, $a = b \neq 0$
(ii) Coefficient of xy is zero, *i.e.* $h = 0$.

(4) The equation of the circle through three non-collinear points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

(5) The point $P(x_1, y_1)$ lies outside, on or inside the circle

$$S \equiv x^2+y^2+2gx+2fy+c = 0, \text{ according as}$$

$$S_1 = x_1^2+y_1^2+2gx_1+2fy_1+c > = \text{ or } < 0.$$

(6) The parametric co-ordinates of any point on the circle

$$(x-h)^2+(y-k)^2 = r^2 \text{ are given by}$$

$$(h+r\cos\theta, k+r\sin\theta).\ (0 < \theta < 2\pi)$$

In particular co-ordinates of any point on the circle $x^2+y^2 = r^2$ are $(r\cos\theta, r\sin\theta)$. $(0 < \theta < 2\pi)$

(7) Different forms of the equations of a circle :

(i) $(x-r)^2+(y-r)^2 = r^2$ is the equation of circle with centre (r, r), radius r and touches both the axes.
(ii) $(x-x_1)^2+(y-r)^2 = r^2$ is the equation of circle with centre (x_1, r), radius r and touches x-axis only.
(iii) $(x-r)^2+(y-y_1)^2 = r^2$ is the equation of circle with centre (r, y_1), radius r and touches y-axis only.
(iv) $x^2+y^2-\alpha x-\beta y = 0$ is the equation of circle with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$, radius $\sqrt{\left(\frac{\alpha^2+\beta^2}{4}\right)}$ which passes through the origin (0, 0) and has intercepts α and β on the axis of X and Y respectively.

(8) The equation of the **tangent** to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ and that of the normal is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

In particular, the equation of tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) is $xx_1 + yy_1 = r^2$ and that of the normal $\frac{x}{x_1} = \frac{y}{y_1}$

Note : Normal to a circle passes through its centre.

(9) The general equation of a line with slope m and which is tangent to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$(y + f) = m(x + g) \pm \sqrt{(g^2 + f^2 - c)}\sqrt{(1 + m^2)}$$

In particular, the equation of the tangent to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{(1 + m^2)}$. If m is infinite, then the tangents are $x \pm a = 0$.

(10) The locus of point of intersection of two perpendicular tangents is called the **director circle.** The director circle of the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

(11) Equation of the chord of the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

in terms of the middle point (x_1, y_1) is

$$T = S_1$$

where $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

In particular, equation of the chord of the circle

$$x^2 + y^2 = a^2$$

in terms of the middle point (x_1, y_1) is

$$xx_1 + yy_1 = x_1^2 + y_1^2$$

(12) Equation of the Chord of Contact :

Equation of the chord of contact of the circle

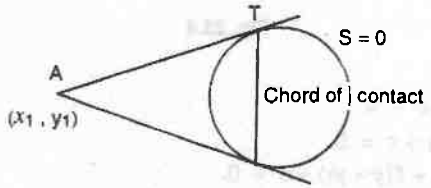


Fig. 23.1

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

which is designated by $T = 0$.

(13) Length of tangent :

$$l(AT) = \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)} = \sqrt{S_1}$$

(14) Equation of the circles given in diagram are :

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$$
$$+ \cot\theta\{(y - y_2)(x - x_1) - (x - x_2)(y - y_1)\} = 0$$

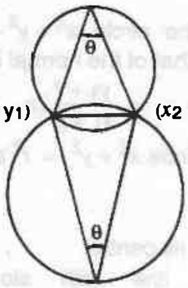


Fig. 23.2

(15) Orthogonality of two circles :

In ΔPC_1C_2 (Fig. 23.3)

$$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2$$

$$d^2 = r_1^2 + r_2^2$$

$$(g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

$$\Rightarrow \quad 2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

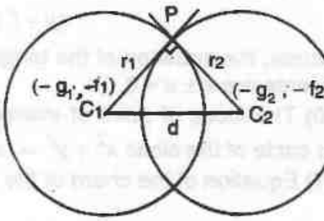


Fig. 23.3

(16) Pair of tangents :

Tangents are drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ (Fig. 23.4) then equation of pair of tangents is

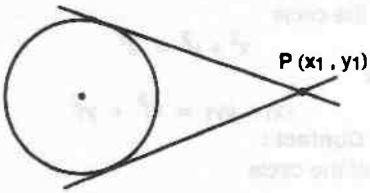


Fig. 23.4

$$SS_1 = T^2$$

where $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

(17) Equation of straight line PQ joining two points θ and φ on the circle $x^2 + y^2 = a^2$ is

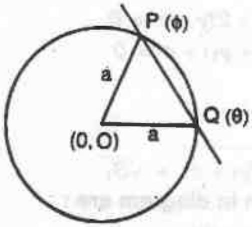


Fig. 23.5

$$x \cos\left(\frac{\theta + \phi}{2}\right) + y \sin\left(\frac{\theta + \phi}{2}\right) = a \cos\left(\frac{\phi - \theta}{2}\right)$$

(18) External and Internal Contacts of Circles :

Two circles with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1, r_2 respectively touch each other.

(i) **Externally :** If $|C_1C_2| = r_1 + r_2$ and the point of contact

$$\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

(ii) **Internally :** If $|C_1C_2| = |r_1 - r_2|$ and the point of contact is

$$\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$$

(19) Common tangents :

Find T using

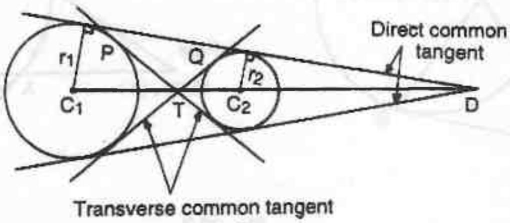


Fig. 23.6

$$\frac{C_1T}{C_2T} = \frac{r_1}{r_2}$$

and find D,

$$\frac{C_1D}{C_2D} = \frac{r_1}{r_2}$$

To find equations of common tangents :

Now assume the equation of tangent of any circle in the form of the slope

$$(y + f) = m(x + g) + a\sqrt{(1 + m^2)}$$

(where a is the radius of the circle)

T and D will satisfy the assumed equation. Thus obtained 'm'. We can find the equation of common tangent if substitute the value of m in the assumed equation.

(20) (i) The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

where λ is a parameter.

(ii) $(x - x_1)^2 + (y - y_1)^2 + \lambda[(y - y_1) - m(x - x_1)] = 0$ is the family of circles which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m. If m is infinite, the family is $(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$.

(21) Radical axis : The equation of **radical-axis** of two circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 - S_2 = 0$ (coefficient of x^2, y^2 in S_1 & S_2 are 1).

(22) Radical Centre : The common point of intersection of the radical axes of three circles taken two at a time is called the **radical centre** of the three circles.

(23) Pole and Polar : Let $P(x_1, y_1)$ be any point inside or outside the circle. Draw chords AB and $A'B'$ passing through P.

If tangents to the circle at A and B meet at $Q(h, k)$, then locus of Q is called polar of P w.r.t. circle and P is called the pole and if tangents to the circle at A' and B' meet at Q', then the straight line QQ' is polar with P' as its pole.

Hence equation of polar of $P(x_1, y_1)$ with respect to $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2 \quad \text{or } (T = 0\ (x_1, y_1))$$

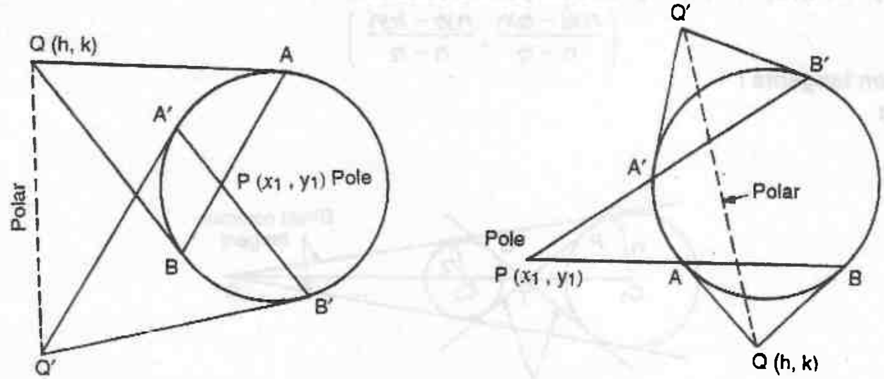


Fig. 23.7

(24) Family of Circle :

Let
$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$
$$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$$
and
$$L = px + qy + r = 0, \quad \text{then}$$

(i) If $S = 0$ and $S' = 0$ intersect in real and distinct points, $S + \lambda S' = 0$ $(\lambda \neq -1)$ represents **a family of circles** passing through these points. $S - S' = 0$ (for $\lambda = -1$) represents the **common chord** of the circles $S = 0$ and $S' = 0$.

(ii) $S = 0$ and $S' = 0$ touch each other, $S - S' = 0$ is the equation of the **common tangent** to the two circles at the point of contact.

(iii) If $S = 0$ and $L = 0$ intersect in two real distinct points, $S + \lambda L = 0$ represents a family of circles passing through these points.

(iv) If $L = 0$ is a tangent to the circle $S = 0$ at P, $S + \lambda L = 0$ represents a family of circles touching $S = 0$ at P having $L = 0$ as the common tangent at P.

(25) Co-axial family of circles : A system of circles is said to be co-axial if every pair of circles of this family has the same radical axis.

The equation of co-axial system is

$$x^2 + y^2 + 2gx + c = 0$$

where g is parameter and c is constant. The equation of other family of co-axial circles is

$$x^2 + y^2 + 2fy + c = 0$$

where f is parameter and c is constant.

(26) Limiting points of Co-axial system of Circles :

Point circles : Circles whose radii are zero are called point circles.

Limiting points of a system of co-axial circles are the centres of the point circles belonging to the family.

Two such points of a co-axial system are $(\pm \sqrt{c}, 0)$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letter a, b, c, d whichever is appropriate.

1. The number of rational point(s) (a point (a, b) is rational, if a and b both are rational numbers) on the circumference of a circle having centre (π, e) is
(a) at most one (b) at least two
(c) exactly two (d) infinite

2. The locus of a point such that the tangents drawn from it to the circle $x^2 + y^2 - 6x - 8y = 0$ are perpendicular to each other is
(a) $x^2 + y^2 - 6x - 8y - 25 = 0$
(b) $x^2 + y^2 + 6x - 8y - 5 = 0$
(c) $x^2 + y^2 - 6x + 8y - 5 = 0$
(d) $x^2 + y^2 - 6x - 8y + 25 = 0$

3. The locus of the point $(\sqrt{(3h+2)}, \sqrt{3k})$. If (h, k) lies on $x + y = 1$ is
(a) a pair of straight lines
(b) a circle
(c) a parabola
(d) an ellipse

4. The equation of the pair of straight lines parallel to the y-axis and which are tangents to the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ is
(a) $x^2 - 4x - 21 = 0$
(b) $x^2 - 5x + 6 = 0$
(c) $x^2 - 6x - 16 = 0$
(d) None of these

5. If a line segment $AM = a$ moves in the plane XOY remaining parallel to OX so that the left end point A slides along the circle $x^2 + y^2 = a^2$, the locus of M is
(a) $x^2 + y^2 - 4a^2$
(b) $x^2 + y^2 = 2ax$
(c) $x^2 + y^2 = 2ay$
(d) $x^2 + y^2 - 2ax - 2ay = 0$

6. If (2, 5) is an interior point of the circle $x^2 + y^2 - 8x - 12y + p = 0$ and the circle neither cuts nor touches any one of the axes of co-ordinates then
(a) $p \in (36, 47)$ (b) $p \in (16, 47)$
(c) $p \in (16, 36)$ (d) None of these

7. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then
(a) $a_1b_1 = a_2b_2$
(b) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$
(c) $a_1 + a_2 = b_1 + b_2$
(d) $a_1a_2 = b_1b_2$

8. AB is a diameter of a circle and C is any point on the circumference of the circle. Then
(a) The area of ΔABC is maximum when it is isosceles
(b) The area of ΔABC is minimum when it is isosceles
(c) The perimeter of ΔABC is maximum when it is isosceles
(d) None of these

9. The four points of intersection of the lines $(2x - y + 1)(x - 2y + 3) = 0$ with the axes lie on a circle whose centre is at the point
(a) (–7/4, 5/4) (b) (3/4, 5/4)
(c) (9/4, 5/4) (d) (0, 5/4)

10. Origin is a limiting point of a co-axial system of which $x^2 + y^2 - 6x - 8y + 1 = 0$ is a member. The other limiting point is
(a) $(-2, -4)$ (b) $\left(\frac{3}{25}, \frac{4}{25}\right)$
(c) $\left(-\frac{3}{25}, -\frac{4}{25}\right)$ (d) $\left(\frac{4}{25}, \frac{3}{25}\right)$

11. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is
(a) $4 \le x^2 + y^2 \le 64$
(b) $x^2 + y^2 \le 25$
(c) $x^2 + y^2 > 25$
(d) $3 < x^2 + y^2 < 9$

12. Three sides of a triangle have the equations $L_r = y - m_r x - C_r = 0$; $r = 1, 2, 3$. Then $\lambda L_2 L_3 + \mu L_3 L_1 + \gamma L_1 L_2 = 0$, where $\lambda \neq 0, \mu \neq 0, \gamma \neq 0$, is the equation of circumcircle of triangle if

(a) $\lambda (m_2 + m_3) + \mu (m_3 + m_1) + \gamma (m_1 + m_2) = 0$

(b) $\lambda (m_2 m_3 - 1) + \mu (m_3 m_1 - 1) + \gamma (m_1 m_2 - 1) = 0$

(c) both (a) & (b)

(d) None of these

13. The abscissaes of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. The radius of the circle with AB as diameter is

(a) $\sqrt{(a^2 + b^2 + p^2 + q^2)}$

(b) $\sqrt{(a^2 + p^2)}$

(c) $\sqrt{(b^2 + q^2)}$

(d) None of these

14. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g_1 x + 2f_1 y = 0$ touch each other, then

(a) $f_1 g = f g_1$ (b) $f f_1 = g g_1$

(c) $f^2 + g^2 = f_1^2 + g_1^2$ (d) None of these

15. The number of integral values of λ for which

$$x^2 + y^2 + \lambda x + (1 - \lambda) y + 5 = 0$$

is the equation of a circle whose radius cannot exceed 5, is

(a) 14 (b) 18

(c) 16 (d) None of these

16. A triangle is formed by the lines whose combined equation is given by $(x + y - 4)(xy - 2x - y + 2) = 0$. The equation of its circum-circle is

(a) $x^2 + y^2 - 5x - 3y + 8 = 0$

(b) $x^2 + y^2 - 3x - 5y + 8 = 0$

(c) $x^2 + y^2 + 2x + 2y - 3 = 0$

(d) None of these

17. The circle $x^2 + y^2 + 4x - 7y + 12 = 0$ cuts an intecept on y-axis of length

(a) 3 (b) 4

(c) 7 (d) 1

18. If a chord of the circle $x^2 + y^2 = 8$ makes equal intercepts of length a on the co-ordinate axes, then

(a) $|a| < 8$ (b) $|a| < 4\sqrt{2}$

(c) $|a| < 4$ (d) $|a| > 4$

19. One of the diameter of the circle $x^2 + y^2 - 12x + 4y + 6 = 0$ is given by

(a) $x + y = 0$ (b) $x + 3y = 0$

(c) $x = y$ (d) $3x + 2y = 0$

20. The coordinates of the middle point of the chord cut off by $2x - 5y + 18 = 0$ by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ are

(a) (1, 4) (b) (2, 4)

(c) (4, 1) (d) (1, 1)

21. Let $\phi(x, y) = 0$ be the equation of a circle. If $\phi(0, \lambda) = 0$ has equal roots $\lambda = 2, 2$ and $\phi(\lambda, 0) = 0$ has roots $\lambda = \frac{4}{5}, 5$ then the centre of the circle is

(a) (2, 29/10) (b) (29/10, 2)

(c) (–2, 29/10) (d) None of these

22. Two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, where $pq \neq 0$, are bisected by the x-axis. Then

(a) $|p| = |q|$ (b) $p^2 = 8q^2$

(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

23. The locus of a point which moves such that the tangents from it to the two circles

$$x^2 + y^2 - 5x - 3 = 0$$

and $3x^2 + 3y^2 + 2x + 4y - 6 = 0$ are equal, is

(a) $2x^2 + 2y^2 + 7x + 4y - 3 = 0$

(b) $17x + 4y + 3 = 0$

(c) $4x^2 + 4y^2 - 3x + 4y - 9 = 0$

(d) $13x - 4y + 15 = 0$

24. The locus of the point of intersection of the tangents to the circle $x = r\cos\theta$, $y = r\sin\theta$ at points whose parametric angles differ by $\pi/3$ is

(a) $x^2 + y^2 = 4(2 - \sqrt{3}) r^2$

(b) $3(x^2 + y^2) = 1$

(c) $x^2 + y^2 = (2 - \sqrt{3}) r^2$

(d) $3(x^2 + y^2) = 42$.

25. If one circle of a co-axial system is $x^2+y^2+2gx+2fy+c=0$ and one limiting point is (a, b) then equation of the radical axis will be

(a) $(g+a)x+(f+b)y+c-a^2-b^2=0$
(b) $2(g+a)x+2(f+b)y+c-a^2-b^2=0$
(c) $2gx+2fy+c-a^2-b^2=0$
(d) None of these

26. $S=x^2+y^2+2x+3y+1=0$ and $S'=x^2+y^2+4x+3y+2=0$ are two circles. The point $(-3,-2)$ lies

(a) inside S' only (b) inside S only
(c) inside S and S' (d) outside S and S'

27. The centre of the circle $r^2=2-4r\cos\theta+6r\sin\theta$ is

(a) (2, 3) (b) (−2, 3)
(c) (−2, −3) (d) (2, −3)

28. If $(1+\alpha x)^n=1+8x+24x^2+\ldots$ and a line through $P(\alpha, n)$ cuts the circle $x^2+y^2=4$ in A and B, then $PA.PB=$

(a) 4 (b) 8
(c) 16 (d) 32

29. One of the diameter of the circle circumscribing the rectangle $ABCD$ is $4y=x+7$. If A and B are the points $(-3, 4)$ and $(5, 4)$ respectively, then the area of the rectangle is

(a) 16 sq. units (b) 24 sq. units
(c) 32 sq. units (d) None of these

30. To which of the following circles, the line $y-x+3=0$ is normal at the point $\left(3+\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$?

(a) $\left(x-3-\frac{3}{\sqrt{2}}\right)^2+\left(y-\frac{3}{\sqrt{2}}\right)^2=9$
(b) $\left(x-\frac{3}{\sqrt{2}}\right)^2+\left(y-\frac{3}{\sqrt{2}}\right)^2=9$
(c) $x^2+(y-3)^2=9$
(d) $(x-3)^2+y^2=9$

31. A circle of radius 5 units touches both the axes and lies in the first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, then its equation in the new position is

(a) $x^2+y^2+20\pi x-10y+100\pi^2=0$
(b) $x^2+y^2+20\pi x+10y+100\pi^2=0$
(c) $x^2+y^2-20\pi x-10y+100\pi^2=0$
(d) None of these

32. If the abscissaes and ordinates of two points P and Q are the roots of the equations $x^2+2ax-b^2=0$ and $x^2+2px-q^2=0$ respectively, then equation of the circle with PQ as diameter is

(a) $x^2+y^2+2ax+2py-b^2-q^2=0$
(b) $x^2+y^2-2ax-2py+b^2+q^2=0$
(c) $x^2+y^2-2ax-2py-b^2-q^2=0$
(d) $x^2+y^2+2ax+2py+b^2+q^2=0$

33. If two circles $(x-1)^2+(y-3)^2=r^2$ and $x^2+y^2-8x+2y+8=0$ intersect in two distinct points, then

(a) $2<r<8$ (b) $r<2$
(c) $r=2$ (d) $r>2$

34. A variable circle always touches the line $y=x$ and passes through the point (0, 0). The common chords of above circle and $x^2+y^2+6x+8y-7=0$ will pass through a fixed point whose coordinates are

(a) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (b) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$
(c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) None of these

35. The locus of the centres of the circles which cut the circles $x^2+y^2+4x-6y+9=0$ and $x^2+y^2-5x+4y+2=0$ orthogonally is

(a) $3x+4y-5=0$
(b) $9x-10y+7=0$
(c) $9x+10y-7=0$
(d) $9x-10y+11=0$

36. If from any point on the circle $x^2+y^2+2gx+2fy+c=0$ tangents are drawn to the circle $x^2+y^2+2gx+2fy+c\sin^2\alpha+(g^2+f^2)\cos^2\alpha=0$, then the angle between the tangents is

(a) 2α (b) α
(c) $\alpha/2$ (d) $\alpha/4$

37. The equations of the circles which touch both the axes and the line $x=a$ are

(a) $x^2+y^2 \pm ax \pm ay + \frac{a^2}{4} = 0$

(b) $x^2+y^2 + ax \pm ay + \frac{a^2}{4} = 0$

(c) $x^2+y^2 - ax + ay + \frac{a^2}{4} = 0$

(d) None of these

38. A, B, C and D are the points of intersection with the coordinate axes of the lines $ax+by=ab$ and $bx+ay=ab$, then

(a) A, B, C, D are concyclic

(b) A, B, C, D form a parallelogram

(c) A, B, C, D form a rhombus

(d) None of these

39. The common chord of $x^2+y^2-4x-4y=0$ and $x^2+y^2=16$ subtends at the origin an angle equal to

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

40. The number of common tangents that can be drawn to the circles $x^2+y^2-4x-6y-3=0$ and $x^2+y^2=2x+2y+1=0$ is

(a) 1 (b) 2

(c) 3 (d) 4

41. If the distances from the origin of the centres of three circles $x^2+y^2+2\lambda_i x - c^2 = 0$ $(i=1, 2, 3)$ are in G.P., then the lengths of the tangents drawn to them from any point on the circle $x^2+y^2=c^2$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

42. If $4l^2-5m^2+6l+1=0$ and the line $lx+my+1=0$ touches a fixed circle, then

(a) the centre of the circle is at the point (4, 0)

(b) the radius of the circle is equal to $\sqrt{5}$

(c) the circle passes through origin

(d) None of these

43. A variable chord is drawn through the origin to the circle $x^2+y^2-2ax=0$. The locus of the centre of the circle drawn on this chord as diameter is

(a) $x^2+y^2+ax=0$ (b) $x^2+y^2+ay=0$

(c) $x^2+y^2-ax=0$ (d) $x^2+y^2-ay=0$

44. If a circle passes through the point (a, b) and cuts the circle $x^2+y^2=\lambda^2$ orthogonally, equation of the locus of its centre is

(a) $2ax+2by=a^2+b^2+\lambda^2$

(b) $ax+by=a^2+b^2+\lambda^2$

(c) $x^2+y^2+2ax+2by+\lambda^2=0$

(d) $x^2+y^2-2ax-2by+a^2+b^2-\lambda^2=0$

45. If O is the origin and OP, OQ are distinct tangents to the circle $x^2+y^2+2gx+2fy+c=0$, the circumcentre of the triangle OPQ is

(a) $(-g, -f)$ (b) (g, f)

(c) $(-f, -g)$ (d) None of these

46. The circle passing through the distinct points $(1, t)$, $(t, 1)$ and (t, t) for all values of t, passes through the point

(a) (1, 1) (b) (–1, –1)

(c) (1, –1) (d) (–1, 1).

47. Equation of a circle through the origin and belonging to the co-axial system, of which the limiting points are (1, 2), (4, 3) is

(a) $x^2+y^2-2x+4y=0$

(b) $x^2+y^2-8x-6y=0$

(c) $2x^2+2y^2-x-7y=0$

(d) $x^2+y^2-6x-10y=0$

48. Equation of the normal to the circle $x^2+y^2-4x+4y-17=0$ which passes through (1, 1) is

(a) $3x+2y-5=0$

(b) $3x+y-4=0$

(c) $3x+2y-2=0$

(d) $3x-y-8=0$

49. α, β and γ are parametric angles of three points P, Q and R respectively, on the circle $x^2+y^2=1$ and A is the point $(-1, 0)$. If the lengths of the chords AP, AQ and AR are in G.P., then $\cos\alpha/2$, $\cos\beta/2$ and $\cos\gamma/2$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

50. The area bounded by the circles $x^2+y^2=r^2$, $r=1, 2$ and the rays given by $2x^2-3xy-2y^2=0$, $y>0$ is

(a) $\frac{\pi}{4}$ sq. units (b) $\frac{\pi}{2}$ sq. units
(c) $\frac{3\pi}{4}$ sq. units (d) π sq. units

51. The equation of the circle touching the lines $|y| = x$ at a distance $\sqrt{2}$ units from the origin is
(a) $x^2 + y^2 - 4x + 2 = 0$
(b) $x^2 + y^2 + 4x - 2 = 0$
(c) $x^2 + y^2 + 4x + 2 = 0$
(d) None of these

52. The values of λ for which the circle $x^2 + y^2 + 6x + 5 + \lambda(x^2 + y^2 - 8x + 7) = 0$ dwindles into a point are
(a) $1 \pm \frac{\sqrt{2}}{3}$
(b) $2 \pm \frac{2\sqrt{2}}{3}$
(c) $2 \pm \frac{4\sqrt{2}}{3}$
(d) $1 \pm \frac{4\sqrt{2}}{3}$

53. The equation of the circle passing through (2, 0) and (0, 4) and having the minimum radius is
(a) $x^2 + y^2 = 20$
(b) $x^2 + y^2 - 2x - 4y = 0$
(c) $(x^2 + y^2 - 4) + \lambda(x^2 + y^2 - 16) = 0$
(d) None of these

54. The shortest distance from the point $(2, -7)$ to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is
(a) 1 (b) 2
(c) 3 (d) 4

55. The circle $x^2 + y^2 = 4$ cuts the line joining the points A (1, 0) and B (3, 4) in two points P and Q. Let $\frac{BP}{PA} = \alpha$ and $\frac{BQ}{QA} = \beta$ then α and β are roots of the quadratic equation
(a) $x^2 + 2x + 7 = 0$
(b) $3x^2 + 2x - 21 = 0$
(c) $2x^2 + 3x - 27 = 0$
(d) None of these

56. The equation of the image of the circle $(x - 3)^2 + (y - 2)^2 = 1$ by the mirror $x + y = 19$ is
(a) $(x - 14)^2 + (y - 13)^2 = 1$
(b) $(x - 15)^2 + (y - 14)^2 = 1$
(c) $(x - 16)^2 + (y - 15)^2 = 1$
(d) $(x - 17)^2 + (y - 16)^2 = 1$

57. A variable circle always touches the line $y = x$ and passes through the point (0, 0). The common chords of above circle and $x^2 + y^2 + 6x + 8y - 7 = 0$ will pass through a fixed point, whose coordinates are
(a) (1, 1) (b) (1/2, 1/2)
(c) (–1/2, –1/2) (d) None of these

58. If P and Q are two points on the circle $x^2 + y^2 - 4x - 4y - 1 = 0$ which are farthest and nearest respectively from the point (6, 5) then
(a) $P \equiv \left(-\frac{22}{5}, 3\right)$
(b) $Q = \left(\frac{22}{5}, \frac{19}{5}\right)$
(c) $P = \left(\frac{14}{3}, -\frac{11}{5}\right)$
(d) $Q \equiv \left(-\frac{14}{3}, -4\right)$

59. If α, β are the roots of $ax^2 + bx + c = 0$ and α', β' those of $a'x^2 + b'x + c' = 0$, the equation of the circle having $A(\alpha, \alpha')$ and $B(\beta, \beta')$ as diameter is
(a) $cc'(x^2 + y^2) + ac'x + a'cy + a'b + ab' = 0$
(b) $cc'(x^2 + y^2) + a'cx + ac'y + a'b + ab' = 0$
(c) $bb'(x^2 + y^2) + a'bx + ab'y + a'c + ac' = 0$
(d) $aa'(x^2 + y^2) + a'bx + ab'y + a'c + ac' = 0$

60. The circle $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ touch each other then
(a) $a = b \pm 2c$
(b) $a = b \pm \sqrt{2}\,c$
(c) $a = b \pm c$
(d) None of these

MULTIPLE CHOICE –II

Each question in this part has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

61. The equation of the circle which touches the axis of coordinates and the line $\frac{x}{3}+\frac{y}{4}=1$ and whose centre lies in the first quadrant is $x^2+y^2-2\lambda x-2\lambda y+\lambda^2=0$, where λ is equal to
(a) 1 (b) 2
(c) 3 (d) 6

62. If P is a point on the circle $x^2+y^2=9$, Q is a point on the line $7x+y+3=0$, and the line $x-y+1=0$ is the perpendicular bisector of PQ, then the coordinates of P are
(a) (3, 0) (b) $\left(\frac{72}{25}, -\frac{21}{25}\right)$
(c) (0, 3) (d) $\left(-\frac{72}{25}, \frac{21}{25}\right)$

63. If a circle passes through the point $\left(3, \sqrt{\frac{7}{2}}\right)$ and touches $x+y=1$ and $x-y=1$, then the centre of the circle is
(a) (4, 0) (b) (4, 2)
(c) (6, 0) (e) (7, 9)

64. The tangents drawn from the origin to the circle $x^2+y^2-2px-2qy+q^2=0$ are perpendicular if
(a) $p=q$ (b) $p^2=q^2$
(c) $q=-p$ (d) $p^2+q^2=1$

65. An equation of a circle touching the axes of coordinates and the line $x\cos\alpha+y\sin\alpha=2$ can be
(a) $x^2+y^2-2gx-2gy+g^2=0$ where $g=2/(\cos\alpha+\sin\alpha+1)$
(b) $x^2+y^2-2gx-2gy+g^2=0$ where $g=2/(\cos\alpha+\sin\alpha-1)$
(c) $x^2+y^2-2gx+2gy+g^2=0$ where $g=2/(\cos\alpha-\sin\alpha+1)$
(d) $x^2+y^2-2gx+2gy+g^2=0$ where $g=2/(\cos\alpha-\sin\alpha-1)$

66. Equation of the circle cutting orthogonally the three circles $x^2+y^2-2x+3y-7=0, x^2+y^2+5x-5y+9=0$ and $x^2+y^2+7x-9y+29=0$ is
(a) $x^2+y^2-16x-18y-4=0$
(b) $x^2+y^2-7x+11y+6=0$
(c) $x^2+y^2+2x-8y+9=0$
(d) None of these

67. A line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2+y^2=r^2$ at A and B, then $PA.PB$ is equal to
(a) $(\alpha+\beta)^2-r^2$ (b) $\alpha^2+\beta^2+r^2$
(c) $(\alpha-\beta)^2+r^2$ (d) None of these

68. If α is the angle subtended at $P(x_1, y_1)$ by the circle
$S=x^2+y^2+2gx+2fy+c=0$, then
(a) $\cot\alpha=\frac{\sqrt{S_1}}{\sqrt{g^2+f^2-c}}$
(b) $\cot\alpha/2=\frac{\sqrt{S_1}}{\sqrt{g^2+f^2-c}}$
(c) $\tan\alpha=\frac{2\sqrt{g^2+f^2-c}}{\sqrt{S_1}}$
(d) $\alpha=2\tan^{-1}\left(\frac{\sqrt{g^2+f^2-c}}{\sqrt{S_1}}\right)$

69. The two circles $x^2+y^2+ax=0$ and $x^2+y^2=c^2$ touch each other if
(a) $a+c=0$ (b) $a-c=0$
(c) $a^2=c^2$ (d) None of these

70. The equation of a common tangent to the circles $x^2+y^2+14x-4y+28=0$ and $x^2+y^2-14x+4y-28=0$ is
(a) $x-7=0$
(b) $y-7=0$
(c) $28x+45y+371=0$
(d) $7x-2y+14=0$

71. If A and B are two points on the circle $x^2+y^2-4x+6y-3=0$ which are farthest and nearest respectively from the point (7, 2) then

(a) $A = (2 - 2\sqrt{2}, -3 - 2\sqrt{2})$
(b) $A = (2 + 2\sqrt{2}, -3 + 2\sqrt{2})$
(c) $B = (2 + 2\sqrt{2}, -3 + 2\sqrt{2})$
(d) $B = (2 - 2\sqrt{2}, -3 + 2\sqrt{2})$

72. The equations of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. The radius of a circle touching all the four circles is
(a) $(\sqrt{2} - 1)a$ (b) $2\sqrt{2}a$
(b) $(\sqrt{2} + 1)a$ (d) $(2 + \sqrt{2})a$

73. The equation of a circle C_1 is $x^2 + y^2 = 4$. The locus of the intersection of orthogonal tangents to the circle is the curve C_2 and the locus of the intersection of perpendicular tangents to the curve C_2 is the curve C_3. Then
(a) C_3 is a circle
(b) The area enclosed by the curve C_3 is 8π
(c) C_2 and C_3 are circles with the same centre
(d) None of these

74. The equation of circle passing through (3, –6) and touching both the axes is
(a) $x^2 + y^2 - 6x + 6y + 9 = 0$
(b) $x^2 + y^2 + 6x - 6y + 9 = 0$
(c) $x^2 + y^2 + 30x - 30y + 225 = 0.$
(d) $x^2 + y^2 - 30x + 30y + 215 = 0$

75. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is
(a) 2 (b) 1/3
(c) 6 (d) 3

76. The equation of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, are
(a) $x = 0$
(b) $y = 0$
(c) $(h^2 - r^2)x - 2rhy = 0$
(d) $(h^2 - r^2)x + 2rhy = 0$

77. Equation of a circle with centre (4, 3) touching the circle $x^2 + y^2 = 1$ is
(a) $x^2 + y^2 - 8x - 6y - 9 = 0$
(b) $x^2 + y^2 - 8x - 6y + 11 = 0$
(c) $x^2 + y^2 - 8x - 6y - 11 = 0$
(d) $x^2 + y^2 - 8x - 6y + 9 = 0$

78. The equation of a tangent to the circle $x^2 + y^2 = 25$ passing through $(-2, 11)$ is
(a) $4x + 3y = 25$
(b) $3x + 4y = 38$
(c) $24x - 7y + 125 = 0$
(d) $7x + 24y = 230$

79. The tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular if
(a) $h = r$ (b) $h = -r$
(c) $r^2 + h^2 = 1$ (d) $r^2 = h^2$

80. The equation of the circle which touches the axes of the coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centre lies in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is
(a) 1 (b) 2
(c) 3 (d) 6

81. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts each of the circles $x^2 + y^2 - 4 = 0$, $x^2 + y^2 - 6x - 8y + 10 = 0$ and $x^2 + y^2 + 2x - 4y - 2 = 0$ at the extremities of a diameter, then
(a) $c = -4$
(b) $g + f = c - 1$
(c) $g^2 + f^2 - c = 17$
(d) $gf = 6$

82. A line meets the coordinate axes in A and B. A circle is circumscribed about the triangle OAB. If m and n are the distances of the tangent to the circle at the origin from the points A and B respectively, the diameter of the circle is
(a) $m(m + n)$ (b) $m + n$
(c) $n(m + n)$ (d) $\frac{1}{2}(m + n)$

83. From the point $A(0, 3)$ on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M, such that $AM = 2AB$. An equation of the locus of M is
(a) $x^2 + 6x + (y - 2)^2 = 0$
(b) $x^2 + 8x + (y - 3)^2 = 0$
(c) $x^2 + y^2 + 8x - 6y + 9 = 0$
(d) $x^2 + y^2 + 6x - 4y + 4 = 0$

84. If a chord of the circle $x^2 + y^2 - 4x - 2y - c = 0$ is trisected at the points (1/3, 1/3) and (8/3, 8/3), then

(a) $c = 10$ (b) $c = 20$

(c) $c = 15$ (d) $c^2 - 40c + 400 = 0$

85. The locus of the point of intersection of the lines

$x = a\left(\dfrac{1 - t^2}{1 + t^2}\right)$ and $y = \dfrac{2at}{1 + t^2}$ represents t being a parameter

(a) circle (b) parabola

(c) Ellipse (d) Hyperbola

86. Consider the circles

$$C_1 \equiv x^2 + y^2 - 2x - 4y - 4 = 0$$

and $$C_2 \equiv x^2 + y^2 + 2x + 4y + 4 = 0$$

and the line $L \equiv x + 2y + 2 = 0$. Then

(a) L is the radical axis of C_1 and C_2

(b) L is the common tangent of C_1 and C_2

(c) L is the common chord of C_1 and C_2

(d) L is perp. to the line joining centres of C_1 and C_2

87. If (2, 1) is a limiting point of a co-axial system of circles containing $x^2 + y^2 - 6x - 4y - 3 = 0$, then the other limiting point is

(a) (2, 4) (b) (−5, −6)

(c) (3, 5) (d) (−2, 4)

88. Equation of the circle having diameters $x - 2y + 3 = 0$, $4x - 3y + 2 = 0$ and radius equal to 1 is

(a) $(x - 1)^2 + (y - 2)^2 = 1$

(b) $(x - 2)^2 + (y - 1)^2 = 1$

(c) $x^2 + y^2 - 2x - 4y + 4 = 0$

(d) $x^2 + y^2 - 3x - 4y + 7 = 0$

89. Length of the tangent drawn from any point of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + d = 0$, $(d > c)$ is

(a) $\sqrt{c - d}$ (b) $\sqrt{d - c}$

(c) $\sqrt{g - f}$ (d) $\sqrt{f - g}$

90. A region in the x-y plane is bounded by the curve $y = \sqrt{(25 - x^2)}$ and the line $y = 0$. If the point $(a, a + 1)$ lies in the interior of the region then

(a) $a \in (-4, 3)$

(b) $a \in (-\infty, -1) \cup (3, \infty)$

(c) $a \in (-1, 3)$

(d) None of these

Practice Test

M.M. 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. If point $P(x, y)$ is called a lattice point if $x, y \in I$. Then the total number of lattice points in the interior of the circle $x^2 + y^2 = a^2$, $a \neq 0$ can not be

(a) 202 (b) 203

(c) 204 (d) 205

2. The equation of the circle having its centre on the line $x + 2y - 3 = 0$ and passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 - 4x - 2y + 1 = 0$ is

(a) $x^2 + y^2 - 6x + 7 = 0$

(b) $x^2 + y^2 - 3x + 4 = 0$

(c) $x^2 + y^2 - 2x - 2y + 1 = 0$

(d) $x^2 + y^2 + 2x - 4y + 4 = 0$

3. The locus of a centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis is given by the equation

(a) $x^2 - 6x - 10y + 14 = 0$

(b) $x^2 - 10x - 6y + 14 = 0$

(c) $y^2 - 6x - 10y + 14 = 0$

(d) $y^2 - 10x - 6y + 14 = 0$

4. If $f(x + y) = f(x) . f(y)$ for all x and y, $f(1) = 2$ and $\alpha_n = f(n)$, $n \in N$, then the equation of

the circle having (α_1, α_2) and (α_3, α_4) as the ends of its one diameter is

(a) $(x-2)(x-8)+(y-4)(y-16)=0$
(b) $(x-4)(x-8)+(y-2)(y-16)=0$
(c) $(x-2)(x-16)+(y-4)(y-8)=0$
(d) $(x-6)(x-8)+(y-5)(y-6)=0$

5. A circle of the co-axial system with limiting points (0, 0) and (1, 0) is

(a) $x^2+y^2-2x=0$ (b) $x^2+y^2-6x+3=0$
(c) $x^2+y^2=1$ (d) $x^2+y^2-2x+1=0$

6. If a variable circle touches externally two given circles then the locus of the centre of the variable circle is

(a) a straight line (b) a parabola
(c) an ellipse (d) a hyperbola

7. A square is inscribed in the circle $x^2+y^2-10x-6y+30=0$. One side of the square is parallel to $y=x+3$, then one vertex of the square is

(a) (3, 3)
(b) (7, 3)
(c) $(6, 3-\sqrt{3})$
(d) $(6, 3+\sqrt{3})$

8. The circles $x^2+y^2-4x-81=0$, $x^2+y^2+24x-81=0$ intersect each other at points A and B. A line through point A meet one circle at P and a parallel line through B meet the other circle at Q. Then the locus of the mid point of PQ is

(a) $(x+5)^2+(y+0)^2=25$
(b) $(x-5)^2+(y-0)^2=25$
(c) $x^2+y^2+10x=0$
(d) $x^2+y^2-10x=0$

9. The locus of the mid points of the chords of the circle $x^2+y^2+4x-6y-12=0$ which subtend an angle of $\pi/3$ radians at its circumference is

(a) $(x+2)^2+(y-3)^2=6{\cdot}25$
(b) $(x-2)^2+(y+3)^2=6{\cdot}25$
(c) $(x+2)^2+(y-3)^2=18{\cdot}75$
(d) $(x+2)^2+(y+3)^2=18{\cdot}75$

10. The point $([P+1], [P])$, (where [.] denotes the greatest integer function) lying inside the region bounded by the circle $x^2+y^2-2x-15=0$ and $x^2+y^2-2x-7=0$ then

(a) $P \in [-1, 0) \cup [0, 1) \cup [1, 2)$
(b) $P \in [-1, 2) - \{0, 1\}$
(c) $P \in (-1, 2)$
(d) None of these

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice–I

1. (a)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (b)	**6.** (a)
7. (d)	**8.** (a)	**9.** (a)	**10.** (b)	**11.** (a)	**12.** (c)
13. (a)	**14.** (a)	**15.** (c)	**16.** (b)	**17.** (d)	**18.** (c)
19. (b)	**20.** (a)	**21.** (b)	**22.** (d)	**23.** (b)	**24.** (d)
25. (b)	**26.** (a)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (d)
31. (d)	**32.** (a)	**33.** (a)	**34.** (c)	**35.** (b)	**36.** (a)
37. (c)	**38.** (a)	**39.** (d)	**40.** (c)	**41.** (b)	**42.** (b)

43. (c) **44.** (a) **45.** (d) **46.** (a) **47.** (c) **48.** (b)
49. (b) **50.** (c) **51.** (a) **52.** (c) **53.** (b) **54.** (b)
55. (b) **56.** (d) **57.** (b) **58.** (b) **59.** (d) **60.** (b)

Multiple Choice –II

61. (a), (d) **62.** (a), (d) **63.** (a), (c) **64.** (a), (b), (c) **65.** (a), (b), (c), (d)
66. (a) **67.** (d) **68.** (b), (d) **69.** (a), (b), (c) **70.** (b), (c) **71.** (a), (c)
72. (a), (c) **73.** (a), (c) **74.** (a), (d) **75.** (a), (b) **76.** (a), (c) **77.** (c), (d)
78. (a), (c) **79.** (a), (b) **80.** (a), (d) **81.** (a), (b), (c), (d) **82.** (b)
83. (b), (c) **84.** (b), (d) **85.** (a) **86.** (a), (c), (d) **87.** (b) **88.** (a), (c)
89. (b) **90.** (c)

Practice Test

1. (a), (b), (c) **2.** (a) **3.** (d) **4.** (a) **5.** (d) **6.** (d)
7. (a) (b) **8.** (a) (c) **9.** (a) **10.** (d)

24

CONIC SECTIONS-PARABOLA

§ 24.1 . Conic Sections

It is the locus of a point moving in a plane so that the ratio of its distance from a fixed point (focus) to its distance from a fixed line (directrix) is constant. This ratio is known as *Eccentricity* (denoted by e).

If $e = 1$, then locus is a **Parabola.**

If $e < 1$, then locus is an **Ellipse.**

If $e > 1$, then locus is a **Hyperbola.**

1. Recognisation of Conics :

The equation of conics represented by the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

can be recognise easily by the condition given in the tabular form. For this, first we have to find discriminant of the equation.

We know that the discriminant of above equation is represented by Δ where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

Case I : When $\Delta = 0$,

In this case equation (i) represents the **Degenerate conic** whose nature is given in the following table :

Condition	Nature of Conic
$\Delta = 0$ & $ab - h^2 = 0$	A pair of st. parallel lines or empty set.
$\Delta = 0$ & $ab - h^2 \neq 0$	A pair of intersecting straight lines.
$\Delta = 0$ & $ab < h^2$	Real or Imaginary pair of straight lines.
$\Delta = 0$ & $ab > h^2$	Point.

Case II : When $\Delta \neq 0$,

In this case equation (i) represents the **Non-Degenerate conic** whose nature is given in the following table :

Condition	Nature of conic
$\Delta \neq 0$, $h = 0$, $a = b$	a Circle
$\Delta \neq 0$, $ab - h^2 = 0$	a Parabola
$\Delta \neq 0$, $ab - h^2 > 0$	an Ellipse or empty set.
$\Delta \neq 0$, $ab - h^2 < 0$	a Hyperbola
$\Delta \neq 0$, $ab - h^2 < 0$ and $a + b = 0$	a Rectangular hyperbola.

2. How to find the Centre of Conics :

If $$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Partially Differentiating w.r.t. x and y we get

$$\frac{\partial S}{\partial x} = 2ax + 2hy + 2g, \frac{\partial S}{\partial y} = 2hx + 2by + 2f$$

$$\frac{\partial S}{\partial x} = 2ax + 2hy + 2g, \quad \frac{\partial S}{\partial y} = 2hx + 2by + 2f$$

$$\Rightarrow \qquad ax + hy + g = 0, \quad hx + by + f = 0$$

solving these equations, we get the centre.

$$(x, y) = (x_1, y_1).$$

§ 24.2. Parabola

1. Standard form of Parabola :

The general form of standard parabola is : $y^2 = 4ax$, where a is a constant.

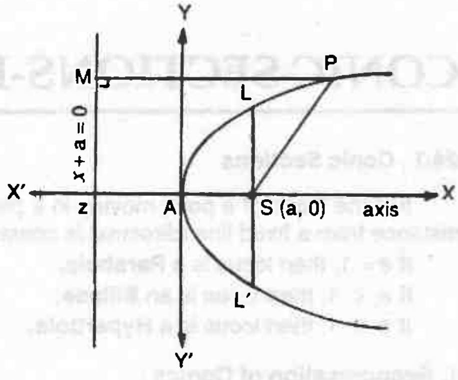


Fig. 24.1

2. Important Properties :

(i) $SP = PM$ and $AS = AZ$

(ii) Vertex is at origin $A = (0, 0)$

(iii) Focus is at $S = (a, 0)$

(iv) Directrix is $x + a = 0$

(v) Axis is $y = 0$ (x-axis).

(vi) Length of latus rectum

$$= LL' = 4a$$

(vii) Ends of the latus rectum are $L = (a, 2a)$ & $L'(a, -2a)$.

(viii) The parametric equation is : $x = at^2$, $y = 2at$.

Note : The other forms of parabola with latus rectum $4a$ are.

(i) $y^2 = -4ax$

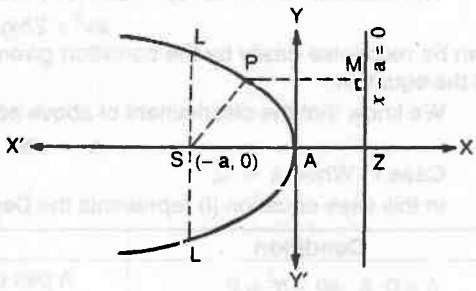


Fig. 24.2

(ii) $x^2 = 4ay$

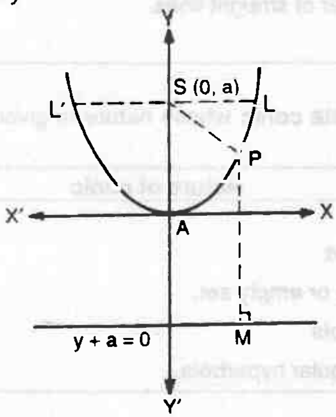


Fig. 24.3

(iii) $x^2 = -4ay$

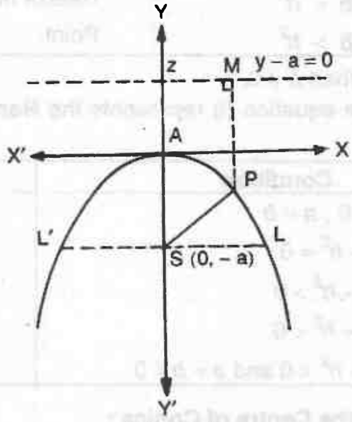


Fig. 24.4

3. General equation of a parabola :

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of the directrix. Let $P(x, y)$ be any point on the parabola. Then by definition.

$\Rightarrow$ $$SP = PM$$

$\Rightarrow$ $$\sqrt{(x-a)^2+(y-b)^2} = \frac{lx+my+n}{\sqrt{l^2+m^2}}$$

$\Rightarrow$ $$(x-a)^2 + (y-b)^2 = \frac{(lx+my+n)^2}{l^2+m^2}$$

$\Rightarrow$ $m^2x^2 + l^2y^2 - 2lmxy + x$ term $+ y$ term + constant $= 0$ This is of the form

$$(mx-ly)^2 + 2gx+2fy+c = 0.$$

This equation is the general equation of parabola.

It should be seen that second degree terms in the general equation of a parabola forms a perfect square.

Note : (i) Equation of the parabola with axis parallel to the x-axis is of the form $\boldsymbol{x = Ay^2 + By + C}$.

(ii) Equation of the parabola with axis parallel to the y-axis is of the form $\boldsymbol{y = Ax^2 + Bx + C}$.

4. Parametric Equations of the Parabola $y^2 = 4ax$

The parametric equations of the parabola $y^2 = 4ax$ are $\boldsymbol{x = at^2}$, $\boldsymbol{y = 2at}$, where t is the parameter. Since the point $(at^2, 2at)$ satisfies the equation $y^2 = 4ax$, therefore the parameteric co-ordiantes of any point on the parabola are $\boldsymbol{(at^2, 2at)}$.

Also the point $(at^2, 2at)$ is reffered as t-point on the parabola.

5. Position of a Point (h, k) with respect to a Parabola $y^2 = 4ax$

Let P be any point (h, k).

Now P will lie **outside, on** or **inside** the parabola according as $\boldsymbol{(k^2 - 4ah) > = < 0}$.

6. Parabola and a Line :

Let the parabola be $y^2 = 4ax$ and the given line be $y = mx + c$.

Hence $\boldsymbol{y = mx + \frac{a}{m}}$, $\boldsymbol{m \neq 0}$ **touches the parabola** $\boldsymbol{y^2 = 4ax}$ **at** $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

7. Equation of the tangent

The equation of the tangent at any point (x_1, y_1) on the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x+x_1)$$

Slope of tangent is $\frac{2a}{y_1}$. (Note)

Corollary 1 : Equation of tangent at any point 't' is

$$ty = x + at^2$$

Slope of tangent is $\frac{1}{t}$.

Corollary 2 : Co-ordinates of the point of intersection of tangents at 't_1' and 't_2' is $\{at_1t_2, a(t_1+t_2)\}$

Corollary 3 : If the chord joining 't_1' and 't_2' to be a focal chord, then $t_1t_2 = -1$.

$\Rightarrow$ $$t_2 = -\frac{1}{t_1}$$

Hence if one extremity of a focal chord is $(at_1^2, 2at_1)$, then the other extremity $(at_2^2, 2at_2)$ **becomes** $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$.

8. Equation of the Normal

The equation of the normal at any point (x_1, y_1) on the parabola $y^2 = 4ax$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

Slope of normal is $-\frac{y_1}{2a}$.

Corollary 1 : Equation of normal at any point 't' is

$$y = -tx + 2at + at^3$$

Slope of normal is $-t$.

Corollary 2 : Co-ordinate of the point of intersection of normals at 't_1' and 't_2' is

$$\{2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2 (t_1 + t_2)\}$$

Corollary 3 : If the normal at the point 't_1' meets the parabola again at 't_2' then

$$t_2 = -t_1 - \frac{2}{t_1}$$

9. Equation of the Normal in Terms of Slope

$$y = mx - 2am - am^3 \text{ at}$$

the point $(am^2, -2am)$

Hence any line $y = mx + c$ will be a normal to the parabola if $c = -2am - am^3$.

10. Equation of chord with mid point (x_1, y_1) :

The equation of the chord of the parabola $y^2 = 4ax$, whose mid point be (x_1, y_1) is

$T = S_1$

where $T = yy_1 - 2a(x + x_1) = 0$

and $S_1 = y_1^2 - 4ax_1 = 0$

11. Chord of contact

If PA and PB be the tangents through point $P(x_1, y_1)$ (Fig. Co. 22) to the parabola $y^2 = 4ax$, then the equation of the chord of contact AB is

$yy_1 = 2a(x + x_1)$ or $T = 0$ (at x_1, y_1)

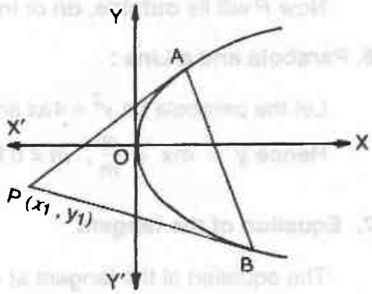


Fig. 24.5

12. Pair of tangents

If $P(x_1, y_1)$ be any point lies outside the parabola $y^2 = 4ax$, and a pair of tangents PA, PB can be draw to it from P, (Fig. Co. 23) then the equation of pair of tangents of PA & PB is

$SS_1 = T^2$

where $S = y^2 - 4ax = 0$

$S_1 = y_1^2 - 4ax_1 = 0$

$T = yy_1 - 2a(x + x_1) = 0.$

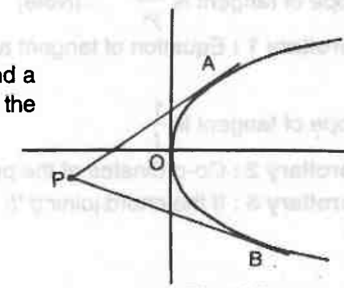


Fig. 24.6

13. Pole and Polar

Let $P(x_1, y_1)$ be any point inside or outside a parabola. Draw chords AB and $A'B'$ passing through P. as shown in Fig. Co. 19.7 (i), (ii)

If tangents to the parabola at A and B meet at $Q(h, k)$, then locus of Q is called polar of P w.r.t. parabola and P is called the pole and if tangents to the parabola at A' and B' meet at Q', then the straight line QQ' is polar with P as its pole.

Hence equation of polar of $P(x_1, y_1)$ with respect to $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

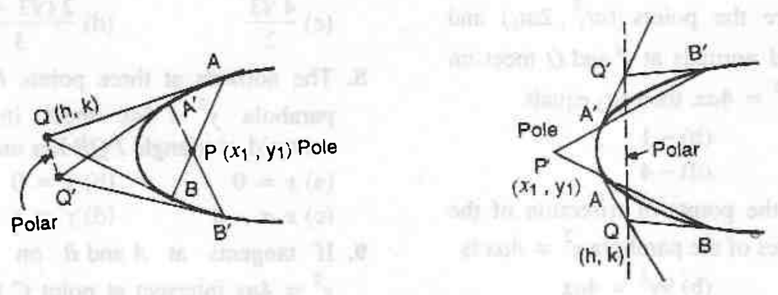


Fig. 24.7

Corollary 1 : Locus of poles of focal chord is $x + a = 0$ *i.e.* directrix or polar of the focus is the directrix.

Corollary 2 : Pole of the chord joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{y_1 y_2}{4a}, \frac{y_1 + y_2}{2}\right)$

Corollary 3 : Any tangent is the polar of its point of contact.

Properties of Pole and Polar :

(i) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$, and such points are said to be **conjugate points.**

(ii) If the pole of a line $ax + by + c = 0$ lies on the another line $a_1x + b_1y + c_1 = 0$ then the pole of the second line will lie on the first and such lines are said to be **conjugate lines.**

14. Diameter :

The locus of the middle points of a system of parallel chords is called a diameter.

If $y = mx + c$ represent a system of parallel chords of the parabola $y^2 = 4ax$ then the line $y = \frac{2a}{m}$ is the equation of the diameter.

15. Reflection Property of a Parabola :

The tangent and normal of the parabola $y^2 = 4ax$ at P are the internal & external bisectors of $\angle SPM$ and BP is parallel to the axis of parabola &

$$\angle BPN = \angle SPN.$$

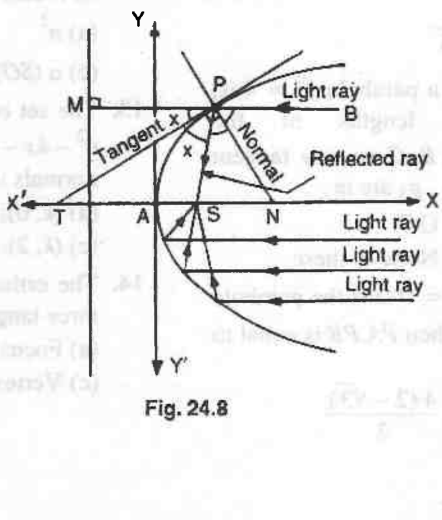


Fig. 24.8

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If P and Q are the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ and normals at P and Q meet on the parabola $y^2 = 4ax$, then t_1t_2 equals

(a) 2 (b) – 1
(c) – 2 (d) – 4

2. The locus of the points of trisection of the double ordinates of the parabola $y^2 = 4ax$ is

(a) $y^2 = ax$ (b) $9y^2 = 4ax$
(c) $9y^2 = ax$ (d) $y^2 = 9ax$

3. If the tangents at P and Q on a parabola meet in T, then SP, ST and SQ are in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

4. If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of ordinates of P and Q is

(a) $4a^2$ (b) $2a^2$
(c) $-4a^2$ (d) $8a^2$

5. The point $(-2m, m+1)$ is an interior point of the smaller region bounded by the circle $x^2+y^2=4$ and the parabola $y^2=4x$. Then m belongs to the interval

(a) $-5-2\sqrt{6} < m < 1$
(b) $0 < m < 4$
(c) $-1 < m < \frac{3}{5}$
(d) $-1 < m < -5+2\sqrt{6}$

6. AB, AC are tangents to a parabola $y^2 = 4ax$, p_1, p_2, p_3 are the lengths of the perpendiculars from A, B, C on any tangent to the curve, then p_2, p_1, p_3 are in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

7. If the line $y-\sqrt{3}x+3 = 0$ cuts the parabola $y^2 = x+2$ at A and B, then $PA.PB$ is equal to [where $P = (\sqrt{3}, 0)$]

(a) $\frac{4(\sqrt{3}+2)}{3}$ (b) $\frac{4(2-\sqrt{3})}{3}$
(c) $\frac{4\sqrt{3}}{2}$ (d) $\frac{2(\sqrt{3}+2)}{3}$

8. The normals at three points P, Q, R of the parabola $y^2 = 4ax$ meet in (h, k). The centroid of triangle PQR lies on

(a) $x = 0$ (b) $y = 0$
(c) $x = -a$ (d) $y = a$

9. If tangents at A and B on the parabola $y^2 = 4ax$ intersect at point C then ordinates of A, C and B are

(a) Always in A.P. (b) Always in G.P.
(c) Always in H.P. (d) None of these

10. The condition that the parabolas $y^2 = 4c(x-d)$ and $y^2 = 4ax$ have a common normal other then x-axis $(a > 0, c > 0)$ is

(a) $2a < 2c+d$ (b) $2c < 2a+d$
(c) $2d < 2a+c$ (d) $2d < 2c+a$

11. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $(y-2)^2 = 4(x+1)$. After reflection, the ray must pass through the point

(a) (–2, 0) (b) (–1, 2)
(c) (0, 2) (d) (2, 0)

12. If the normals at three points, P, Q, R of the parabola $y^2 = 4ax$ meet in a point O and S be its focus, then $|SP|.|SQ|.|SR| =$

(a) a^2 (b) $a(SO)^3$
(c) $a(SO)^2$ (d) None of these

13. The set of points on the axis of the parabola $y^2-4x-2y+5 = 0$ from which all the three normals to the parabola are real is

(a) $(k, 0); k > 1$ (b) $(k, 1); k > 3$
(c) $(k, 2); k > 6$ (d) $(k, 3); k > 8$

14. The orthocentre of a triangle formed by any three tangents to a parabola lies on

(a) Focus (b) Directrix
(c) Vertex (d) Focal chord

15. The vertex of a parabola is the point (a, b) and latus rectum is of length l. If the axis of the parabola is along the positive direction of y-axis then its equation is

(a) $(x-a)^2 = \frac{l}{2}(y-2b)$
(b) $(x-a)^2 = \frac{l}{2}(y-b)$
(c) $(x-a)^2 = l(y-b)$
(d) None of these

16. The equation
$$\sqrt{(x-3)^2+(y-1)^2} + \sqrt{(x+3)^2+(y-1)^2} = 6$$
represents
(a) an ellipse
(b) a pair of straight lines
(c) a circle
(d) a straight line joining the point $(-3, 1)$ to the point $(3, 1)$

17. The condition that the straight line $lx+my+n=0$ touches the parabola $x^2=4ay$ is
(a) $bn = am^2$ (b) $al^2 - mn = 0$
(c) $ln = am^2$ (b) $am = ln^2$

18. The vertex of the parabola whose focus is $(-1, 1)$ and directrix is $4x+3y-24=0$ is
(a) $(0, 3/2)$ (b) $(0, 5/2)$
(c) $(1, 3/2)$ (d) $(1, 5/2)$

19. The slope of a chord of the parabola $y^2=4ax$, which is normal at one end and which subtends a right angle at the origin, is
(a) $1/\sqrt{2}$ (b) $\sqrt{2}$
(c) 2 (d) None of these

20. Let α be the angle which a tangent to the parabola $y^2=4ax$ makes with its axis, the distance between the tangent and a parallel normal will be
(a) $a\sin^2\alpha\cos^2\alpha$ (b) $a\,\text{cosec}\,\alpha\sec^2\alpha$
(c) $a\tan^2\alpha$ (d) $a\cos^2\alpha$

21. If (a, b) is the mid-point of chord passing through the vertex of the parabola $y^2=4x$, then
(a) $a=2b$ (b) $2a=b$
(c) $a^2=2b$ (d) $2a=b^2$

22. The equation to the line touching both the parabolas $y^2=4x$ and $x^2=-32y$ is
(a) $x+2y+4=0$ (b) $2x+y-4=0$
(c) $x-2y-4=0$ (d) $x-2y+4=0$

23. The normal at the point $(at^2, 2at)$ on the parabola $y^2=4ax$ cuts the curve again at the point whose parameter is
(a) $-1/t$ (b) $-\left(t+\frac{2}{t}\right)$
(c) $-2t+\frac{1}{t}$ (d) $t+\frac{2}{t}$

24. If y_1, y_2 are the ordinates of two points P and Q on the parabola and y_3 is the ordinate of the point of intersection of tangents at P and Q then
(a) y_1, y_2, y_3 are in A P. (b) y_1, y_3, y_2 are in A.P.
(c) y_1, y_2, y_3 are in G.P. (d) y_1, y_3, y_2 are in G.P.

25. The equation $ax^2+4xy+y^2+ax+3y+2=0$ represents a parabola if a is
(a) -4 (b) 4
(c) 0 (d) 8

26. The Harmonic mean of the segments of a focal chord of the parabola $y^2=4ax$ is
(a) $4a$ (b) $2a$
(c) a (d) a^2

27. A double ordinate of the parabola $y^2=8px$ is of length 16p. The angel subtended by it at the vertex of the parabola is
(a) $\pi/4$ (b) $\pi/2$
(c) π (d) $\pi/3$

28. The set of points on the axis of the parabola $y^2=4x+8$ from which the 3 normals to the parabola are all real and different is
(a) $\{(k, 0) : k \le -2\}$ (b) $\{(k, 0) : k > -2\}$
(c) $\{(0, k) : k > -2\}$ (d) None of these

29. If $y+b=m_1(x+a)$ and $y+b=m_2(x+a)$ are two tangents to the parabola $y^2=4ax$ then
(a) $m_1+m_2=0$ (b) $m_1m_2=1$
(c) $m_1m_2=-1$ (d) None of these

30. The length of the latus rectum of the parabola $169\{(x-1)^2+(y-3)^2\}=(5x-12y+17)^2$ is
(a) 14/13 (b) 12/13
(c) 28/13 (d) None of these

31. The points on the axis of the parabola $3y^2+4y-6x+8=0$ from when 3 distinct normals can be drawn is given by

(a) $\left(a, \frac{4}{3}\right)$; $a > 19/9$

(b) $\left(a, -\frac{2}{3}\right)$; $a > \frac{19}{9}$

(c) $\left(a, \frac{1}{3}\right)$; $a > \frac{7}{9}$

(d) None of these

32. Let the line $lx+my=1$ cut the parabola $y^2=4ax$ in the points A and B. Normals at A and B meet at point C. Normal from C other than these two meet the parabola at D then the coordinate of D are

(a) $(a, 2a)$ (b) $\left(\frac{4am}{l^2}, \frac{4a}{l}\right)$

(c) $\left(\frac{2am^2}{l^2}, \frac{2a}{l}\right)$ (d) $\left(\frac{4am^2}{l^2}, \frac{4am}{l}\right)$

33. The triangle formed by the tangent to the parabola $y=x^2$ at the point whose abscissa is x_0 ($x_0 \in [1, 2]$), the y-axis and the straight line $y=x_0^2$ has the greatest area if $x_0 =$

(a) 0 (b) 1

(c) 2 (d) 3

34. If the normal at P 't' on $y^2=4ax$ meets the curve again at Q, the point on the curve, the normal at which also passes through Q has co-ordinates (........, ...).

(a) $\left(\frac{2a}{t^2}, \frac{2a}{t}\right)$ (b) $\left(\frac{4a}{t^2}, \frac{2a}{t}\right)$

(c) $\left(\frac{4a}{t^2}, \frac{4a}{t}\right)$ (d) $\left(\frac{4a}{t^2}, \frac{8a}{t}\right)$

35. Two parabolas C and D intersect at two different points, where C is $y=x^2-3$ and D is $y=kx^2$. The intersection at which the x value is positive is designated point A, and $x=a$ at this intersection. the tangent line l at A to the curve D intersects curve C at point B, other than A. If x-value of point B is 1 then $a=$

(a) 1 (b) 2

(c) 3 (d) 4

MULTIPLE CHOICE –II

Each question in this part has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s) :

36. Consider a circle with its centre lying on the focus of the parabola $y^2=2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is

(a) $\left(\frac{p}{2}, p\right)$ (b) $\left(\frac{p}{2}, -p\right)$

(c) $\left(-\frac{p}{2}, p\right)$ (d) $\left(-\frac{p}{2}, -p\right)$

37. The locus of point of intersection of tangents to the parabolas $y^2=4(x+1)$ and $y^2=8(x+2)$ which are perpendicular to each other is

(a) $x+7=0$ (b) $x-y=4$

(c) $x+3=0$ (d) $y-x=12$

38. The equation of a locus is $y^2+2ax+2by+c=0$. Then

(a) it is an ellipse

(b) it is a parabola

(c) its latus rectum $= a$

(d) its latus rectum $= 2a$

39. The equation of a tangent to the parabola $y^2=8x$ which makes an angle 45° with the line $y=3x+5$ is

(a) $2x+y+1=0$ (b) $y=2x+1$

(c) $x-2y+8=0$ (d) $x+2y-8=0$

40. The normal $y=mx-2am-am^3$ to the parabola $y^2=4ax$ subtends a right angle at the vertex if

(a) $m=1$ (b) $m=\sqrt{2}$

(c) $m=-\sqrt{2}$ (d) $m=\frac{1}{\sqrt{2}}$

41. The straight line $x+y=k$ touches the paraboa $y=x-x^2$ if

(a) $k=0$ (b) $k=-1$

(c) $k=1$ (d) k takes any value

42. A tangent to a parabola $y^2 = 4ax$ is inclined at $\pi/3$ with the axis of the parabola. The point of contact is

(a) $(a/3 - 2a/\sqrt{3})$ (b) $(3a, -2\sqrt{3}\,a)$

(c) $(3a, 2\sqrt{3}\,a)$ (d) $(a/3, 2a/\sqrt{3})$

43. If the normals from any point to the parabola $x^2 = 4y$ cuts the line $y = 2$ in points whose abscissae are in A.P., then the slopes of the tangents at the three conormal points are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

44. If the tangent at P on $y^2 = 4ax$ meets the tangent at the vertex in Q, and S is the focus of the parabola, then $\angle SQP =$

(a) $\pi/3$ (b) $\pi/4$

(c) $\pi/2$ (d) $2\pi/3$

45. A focal chord of $y^2 = 4ax$ meets in P and Q. If S is the focus, then $\frac{1}{SP} + \frac{1}{SQ} =$

(a) $\frac{1}{a}$ (b) $\frac{2}{a}$

(c) $\frac{4}{a}$ (d) None of these

46. The diameter of the parabola $y^2 = 6x$ corresponding to the system of parallel chords $3x - y + c = 0$, is

(a) $y - 1 = 0$ (b) $y - 2 = 0$

(c) $y + 1 = 0$ (d) $y + 2 = 0$

47. Let PQ be a chord of the parabola $y^2 = 4x$. A circle drawn with PQ as a diameter passes through the vertex V of the parabola. If Area of $\Delta PVQ = 20$ unit^2 then the co-ordinates of P are

(a) $(-16, -8)$ (b) $(-16, 8)$

(c) $(16, -8)$ (d) $(16, 8)$

48. A line L passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the parabola in two distinct points. If 'm' be the slope of the line L then

(a) $m \in (-1, 1)$

(b) $m \in (-\infty, -1) \cup (1, \infty)$

(c) $m \in R$

(d) None of these

49. The length of the latus rectum of the parabola $x = ay^2 + by + c$ is

(a) $a/4$ (b) $a/3$

(c) $1/a$ (d) $1/4a$

50. P is a point which moves in the x-y plane such that the point P is nearer to the centre of a square than any of the sides. The four vertices of the square are $(\pm a, \pm a)$. The region in which P will move is bounded by parts of parabolas of which one has the equation

(a) $y^2 = a^2 + 2ax$ (b) $x^2 = a^2 + 2ay$

(c) $y^2 + 2ax = a^2$ (d) None of these

Practice Test

M.M : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[$10 \times 2 = 20$]

1. If the normals from any point to the parabola $x^2 = 4y$ cuts the line $y = 2$ in points whose abscissae are in A.P., then the slopes of the tangents at the three co-normal points are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

2. The coordinates of the point on the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$, are

(a) $(2, -4)$ (b) $(18, -12)$

(c) $(2, 4)$ (d) None of these

3. The figure shows the graph of the parabola $y = ax^2 + bx + c$ then

(a) $a > 0$ (b) $b < 0$

(c) $c > 0$ (d) $b^2 - 4ac > 0$

4. The equation of the parabola whose vertex and focus lie on the axis of x at distances a and a_1 from the origin respectively is

(a) $y^2 = 4(a_1 - a)x$

(b) $y^2 = 4(a_1 - a)(x - a)$

(c) $y^2 = 4(a_1 - a)(x - a_1)$

(d) None of these

5. If $(\alpha^2, \alpha - 2)$ be a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, –4) then P belongs to the interval

(a) $-2 + 2\sqrt{2} < \alpha < 2$

(b) $\alpha > -2 + 2\sqrt{2}$

(c) $\alpha > -2 - 2\sqrt{2}$

(d) None of these

6. If a circle and a parabola intersect in 4 points then the algebraic sum of the ordinates is

(a) proportional to arithmatic mean of the radius and latus rectum

(b) zero

(c) equal to the ratio of arithmatic mean and latus rectum

(d) None of these

7. If the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ cuts the parabola again at $(aT^2, 2aT)$ then

(a) $-2 < T < 2$

(b) $T \in (-\infty, -8) \cup (8, \infty)$

(c) $T^2 < 8$

(d) $T^2 > 8$

8. P is the point 't' on the parabola $y^2 = 4ax$ and PQ is a focal chord. PT is the tangent at P and QN is the normal at Q. If the angle between PT and QN be α and the distance between PT and QN be d then

(a) $0 < \alpha < 90°$

(b) $\alpha = 0°$

(c) $d = 0$

(d) $d = a\left(\sqrt{1+t^2} + \dfrac{1}{\sqrt{1+t^2}}\right)$

9. For parabola $x^2 + y^2 + 2xy - 6x - 2y + 3 = 0$, the focus is

(a) (1, – 1) (b) (– 1, 1)

(c) (3, 1) (d) None of these

10. The latus rectum of the parabola

$$x = at^2 + bt + c,\ y = a't^2 + b't + c' \text{ is}$$

(a) $\dfrac{(aa' - bb')^2}{(a^2 + a'^2)^{3/2}}$ (b) $\dfrac{(ab' - a'b)^2}{(a^2 + a'^2)^{3/2}}$

(c) $\dfrac{(bb' - aa')^2}{(b^2 + b'^2)^{3/2}}$ (d) $\dfrac{(a'b - ab')^2}{(b^2 + b'^2)^{3/2}}$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (b)
7. (a)	**8.** (b)	**9.** (a)	**10.** (a)	**11.** (c)	**12.** (c)
13. (b)	**14.** (b)	**15.** (b)	**16.** (d)	**17.** (b)	**18.** (d)
19. (b)	**20.** (b)	**21.** (d)	**22.** (d)	**23.** (b)	**24.** (b)
25. (b)	**26.** (b)	**27.** (b)	**28.** (d)	**29.** (c)	**30.** (c)
31. (b)	**32.** (d)	**33.** (c)	**34.** (c)	**35.** (c)	

Multiple Choice –II

36. (a), (b)	**37.** (c)	**38.** (b), (d)	**39.** (a), (c)	**40.** (b), (c)	**41.** (c)
42. (a), (d)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (a)	**47.** (c), (d)
48. (d)	**49.** (c)	**50.** (a), (b), (c)			

Practice Test

1. (b)	**2.** (a)	**3.** (b,) (c), (d)	**4.** (b)	**5.** (a)	**6.** (b)
7. (d)	**8.** (b)	**9.** (d)	**10.** (b)		

25

ELLIPSE

1. Standard form of an Ellipse :

The general form of standard ellipse is : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, where a & b are constants. Fig. 1.

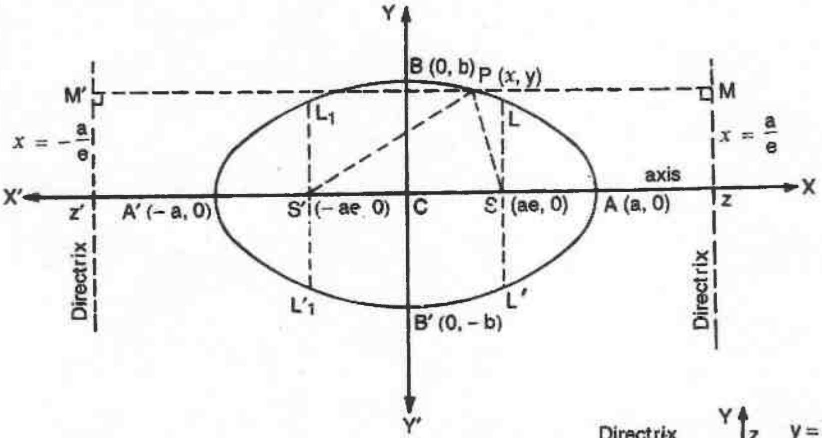


Fig. 1

2. Important properties :

(i) $SP = e\ PM$ and $AS = e\ AZ$

(ii) Co-ordinate of **centre** $C\ (0, 0)$

(iii) $AA' = 2a$ is the **Major axis** of the ellipse.

$BB' = 2b$ is the **Minor axis** of the ellipse.

(iv) Co-ordinates of **vertices** A and A' are $(\pm a, 0)$.

Extremities of minor axis B and B' are $(0, \pm b)$.

(v) Relation in a , b & e is $\mathbf{b^2 = a^2\ (1 - e^2)}$

(vi) Co-ordinates of the **foci** S and S' are $(\pm ae, 0)$.

(vii) Co-ordinates of the **feet of directrices are** $(\pm a/e, 0)$

(viii) Equation of **directrix** $x = \pm a/e$.

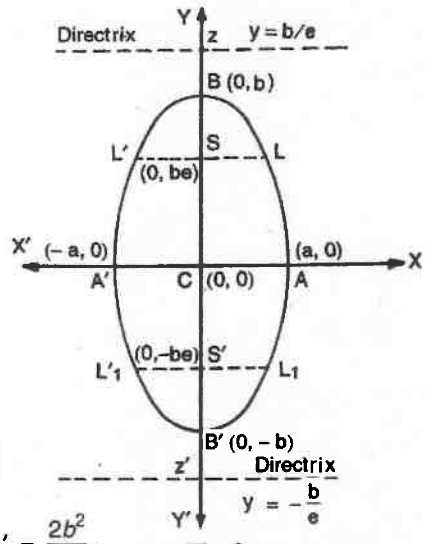


Fig. 2

(ix) Equation of **latus rectum** $\mathbf{x = \pm\ ae}$ and length $LL' = L_1L_1' = \frac{2b^2}{a}$.

(x) **Ends** of the latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$, $L_1\left(-ae, \frac{b^2}{a}\right)$ and $L'_1\left(-ae, -\frac{b^2}{a}\right)$.

(xi) **Focal radii :** $SP = a - ex$ and $S'P = a + ex$

$\Rightarrow$ $SP + S'P = 2a =$ Major axis.

Note : Another form of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a < b)$$

$AA' =$ Minor axis $= 2a$

$BB' =$ Major axis $= 2b$

& $a^2 = b^2 (1 - e^2)$

Latus rectum $LL' = L_1L_1' = \frac{2a^2}{b}$.

3. General Equation of an Ellipse :

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix. Let $P(x, y)$ be any point on the ellipse. Then by definition.

$\Rightarrow$ $SP = e\,PM$ (e is the eccentricity)

$\Rightarrow$ $(x - a)^2 + (y - b)^2 = e^2 \frac{(lx + my + n)^2}{(l^2 + m^2)}$

$\Rightarrow$ $(l^2 + m^2)\{(x - a)^2 + (y - b)^2\} = e^2 (lx + my + n)^2$.

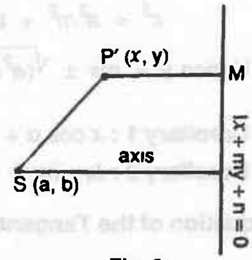


Fig. 3

4. Parametric Equation of the Ellipse :

The parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $x = a\cos\phi$, $y = b\sin\phi$, where ϕ is the parameter. Since the point $(a\cos\phi, b\sin\phi)$ satisfies the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, therefore the parametric co-ordinates of any point on the ellipse is $(a\cos\phi, b\sin\phi)$ also the point $(a\cos\phi, b\sin\phi)$ is reffered as ϕ-point on the ellipse $\phi \in [0, 2\pi)$.

5. Auxiliary Circle and Eccentric angle :

The circle described on the major axis of an ellipse as diameter is called its **auxiliary circle.** (Fig. 4)

The equation of the auxiliary circle is

$$x^2 + y^2 = a^2$$

$\therefore$ $Q = (a\cos\phi, a\sin\phi)$ and $P = (a\cos\phi, b\sin\phi)$

$\phi =$ **eccentric angle.**

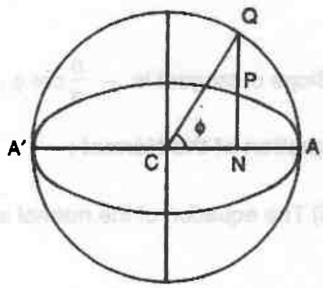


Fig. 4

6. Point and Ellipse :

The point $P(x_1, y_1)$ lies out side, on or inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if

$$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0, = 0, < 0.$$

7. Ellipse and a Line :

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the given line be $y = mx + c$.

Solving the line and ellipse we get

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

i.e., $$(a^2m^2 + b^2)x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$$

above equation being a quadratic in x

$\therefore$ discriminant $= 4m^2c^2a^4 - 4a^2(a^2m^2 + b^2)(c^2 - b^2)$

$= -b^2\{c^2 - (a^2m^2 + b^2)\}$

$= b^2\{(a^2m^2 + b^2) - c^2\}$

Hence the line intersects the parabolas in 2 distinct points if $a^2m^2 + b^2 > c^2$, in one point if $c^2 = a^2m^2 + b^2$, and does not intersect if $a^2m^2 + b^2 < c^2$.

$\therefore$ $y = mx \pm \sqrt{(a^2m^2 + b^2)}$ touches the ellipse and condition for tangency $c^2 = a^2m^2 + b^2$.

Hence $y = mx \pm \sqrt{(a^2m^2 + b^2)}$, touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(\frac{\pm a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right)$

Corollary 1 : $x\cos\alpha + y\sin\alpha = p$ **is a tangent if** $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.

Corollary 2 : $lx + my + n = 0$ **is a tangent if** $n^2 = a^2l^2 + b^2m^2$.

8. Equation of the Tangent :

(i) The equation of the tangent at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

Slope of tangent is $-\frac{b^2x_1}{a^2y_1}$, (Note)

(ii) the equation of tangent at any point 'ϕ' is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

Slope of tangent is $-\frac{b}{a}\cot\phi$.

9. Equation of the Normal :

(i) The equation of the normal at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(ii) The equation of the normal at any point 'ϕ' is

$$ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2$$

10. Equation of Chord Joining two Points *i.e.,* $P(\theta)$ and $Q(\phi)$ is :

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

11. Equation of Chord with Mid point (x_1, y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose mid point be (x_1, y_1) is

$$T = S_1$$

where $$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

$$S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0$$

12. Chord of Contact :

If PA and PB be the tangents through point $P(x_1, y_1)$ (Fig. 5) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of the chord of contact AB is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{or} \quad T = 0 \text{ (at } x_1, y_1\text{)}$$

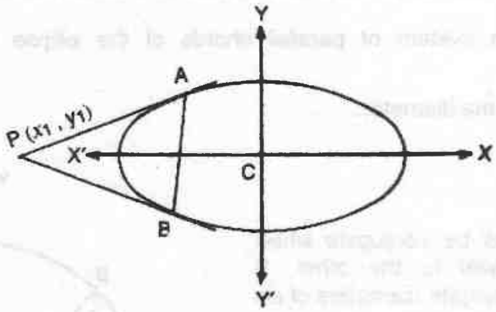


Fig. 5

13. Pair of Tangents :

If $P(x_1, y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

and a pair of tangents PA, PB can be drawn to it from P.

then the equation of pair of tangents of PA & PB is

$$SS_1 = T^2$$

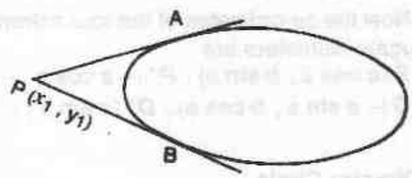


Fig. 6

where

$$S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0$$

$$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

14. Pole and Polar :

Let $P(x_1, y_1)$ be any point inside or outside the ellipse. Draw chords AB and $A'B'$ passing through P,

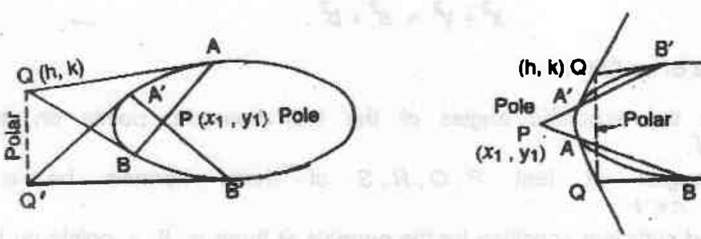


Fig. 7

If tangents to the ellipse at A and B meet at $Q(h, k)$, then locus of Q is called polar of P w.r.t. ellipse and P is called the pole and if tangents to the ellipse at A' and B' meet at Q', then the straight line QQ' is polar with P as its pole. Hence equation of polar of $P(x_1, y_1)$ with respect to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Corollary : The polar of any point on the directrix, passes through the focus.

15. Diameter :

The locus of the middle points of a system of parallel chords is called a diameter.

If $y = mx + c$ represent a system of parallel chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then the line $y = -\frac{b^2}{a^2 m} x$ is the equation of the diameter.

16. Conjugate Diameters :

Two diameters are said to be *conjugate* when each bisects ali chords parallel to the other. If $y = mx$ & $y = m_1 x$ be two conjugate diameters of an ellipse then $mm_1 = -\frac{b^2}{a^2}$.

Conjugate diameters of circle *i.e.* AA' & BB' are perpendicular to each other. Hence conjugate diameters of ellipse are PP' and DD'.

Hence angle between conjugate diameters of ellipse > 90°.

Now the co-ordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi)$, $P'(-a\cos\phi, -b\sin\phi)$

$D(-a\sin\phi, b\cos\phi)$, $D'(a\sin\phi, -b\cos\phi)$

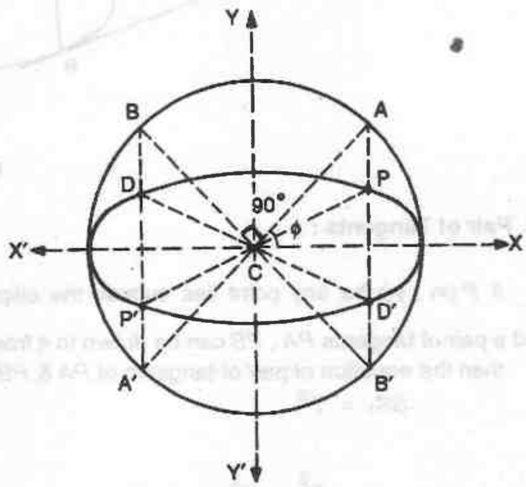


Fig. 8

17. Director Circle :

The locus of the point of intersection of the tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are perpendicular to each other is called **Director circle.**

Hence the equation of director circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = a^2 + b^2.$$

18. Important Conditions of an Ellipse :

(i) If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four concyclic points on an ellipse then $\alpha + \beta + \gamma + \delta = 2n\pi$, $n \in I$.

(ii) If eccentric angles of feet P, Q, R, S of these normals be $\alpha, \beta, \gamma, \delta$ then $\alpha + \beta + \gamma + \delta = (2n+1)\pi$, $n \in I$

(iii) The necessary and sufficient condition for the normals at three α, β, γ points on the ellipse to be concurrent if

$$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0.$$

19. Reflection Property of an Ellipse :

If an incoming light ray passes through one focus (S) strike the concave side of the ellipse then it will get reflected towards other focus (S') & $\angle SPS' = \angle SQS'$.

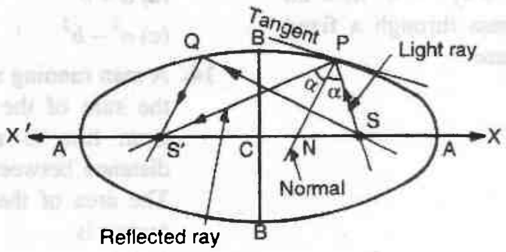


Fig. 9

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. Let P be a variable point on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ with foci at S and S'. If A be the area of triangle PSS', then the maximum value of A is

(a) 24 sq. units (b) 12 sq. units

(c) 36 sq. units (d) None of these

2. The area of a triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle. This ratio is

(a) b/a (b) $2a/b$

(c) a^2/b^2 (d) b^2/a^2

3. The line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if

(a) $\frac{a^2}{m^2} + \frac{b^2}{l^2} = \frac{(a^2 - b^2)^2}{n^2}$

(b) $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$

(c) $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$

(d) None of these

4. The equation $\frac{x^2}{10 - a} + \frac{y^2}{4 - a} = 1$ represents an ellipse if

(a) $a < 4$ (b) $a > 4$

(c) $4 < a < 10$ (d) $a > 10$

5. The set of values of a for which $(13x - 1)^2 + (13y - 2)^2 = a\,(5x + 12y - 1)^2$ represents an ellipse, is

(a) $1 < a < 2$

(b) $0 < a < 1$

(c) $2 < a < 3$

(d) None of these

6. The length of the common chord of the ellipse $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$ and the circle $(x - 1)^2 + (y - 2)^2 = 1$ is

(a) zero (b) one

(c) three (d) eight

7. If CF is the perpendicular from the centre C of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the tangent at any point P, and G is the point when the normal at P meets the major axis, then $CF \cdot PG =$

(a) a^2 (b) ab
(c) b^2 (d) b^3

8. Tangents are drawn from the points on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$, then all the chords of contact pass through a fixed point, whose coordinate are

(a) $\left(\frac{1}{5}, -\frac{2}{5}\right)$
(b) $\left(\frac{4}{5}, -\frac{1}{5}\right)$
(c) $\left(\frac{2}{5}, -\frac{1}{5}\right)$
(d) None of these

9. An ellipse has OB as semi-minor axis. F and F' are its focii and the angle FBF' is a right angle. Then eccentricity of the ellipse is

(a) $1/\sqrt{3}$ (b) $1/2$
(c) $1/\sqrt{2}$ (d) None of these

10. The set of positive value of m for which a line with slope m is a common tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and parabola $y^2 = 4ax$ is given by

(a) (2, 0) (b) (3, 5)
(c) (0, 1) (d) None of these

11. The eccentricity of the ellipse $ax^2 + by^2 + 2fx + 2gy + c = 0$ if axis of ellipse parallel to x-axis is

(a) $\sqrt{\left(\frac{b-a}{b}\right)}$
(b) $\sqrt{\frac{a+b}{b}}$
(c) $\sqrt{\frac{a+b}{a}}$
(d) None of these

12. The minimum length of the intercept of any tangent on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ between the coordinate axes is

(a) $2a$ (b) $2b$
(c) $a - b$ (d) $a + b$

13. If CP and CD are semi-conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $CP^2 + CD^2 =$

(a) $a + b$ (b) $a^2 + b^2$
(c) $a^2 - b^2$ (d) $\sqrt{a^2 + b^2}$

14. A man running round a race course notes that the sum of the distances of two flag-posts from him is always 10 metres and the distance between the flag-posts is 8 metres. The area of the path he encloses in square metres is

(a) 15π (b) 12π
(c) 18π (d) 8π

15. If the normal at the point $P(\phi)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersects it again at the point $Q(2\phi)$, then $\cos\phi$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$
(c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

16. If α and β are eccentric angles of the ends of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ is equal to

(a) $\frac{1-e}{1+e}$ (b) $\frac{e-1}{e+1}$
(c) $\frac{e+1}{e-1}$ (d) $\frac{e-1}{e+3}$

17. The eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose latusrectum is half of its minor axis is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{\frac{2}{3}}$
(c) $\frac{\sqrt{3}}{2}$ (d) None of these

18. The distances from the foci of $P(a, b)$ on the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$ are

(a) $4 \pm \frac{5}{4}b$

(b) $5 \pm \frac{4}{5}a$

(c) $5 \pm \frac{4}{5} b$

(d) None of these

19. If the normal at an end of a latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one extremity of the minor axis, then the eccentricity of the ellipse is given by

(a) $e^4 + e^2 - 1 = 0$

(b) $e^2 + e - 4 = 0$

(c) $e = \sqrt{2}$

(d) $e = 3\sqrt{e}$

20. If A and B are two fixed points and P is a variable point such that $PA + PB = 4$, the locus of P is

(a) a parabola

(b) an ellipse

(c) a hyperbola

(d) None of these

21. The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is

(a) constant and is equal to the product of the axes

(b) can not be constant

(c) constant and is equal to the two lines of the product of the axes

(d) None of these

22. If C be the centre of the ellipse $9x^2 + 16y^2 = 144$ and S is one focus, the ratio of CS to major axis is

(a) $\sqrt{7} : 16$ (b) $\sqrt{7} : 4$

(c) $\sqrt{5} : \sqrt{7}$ (d) None of these

23. The centre of the ellipse $\frac{(x+y-2)^2}{9} + \frac{(x-y)^2}{16} = 1$ is

(a) (0, 0) (b) (1, 1)

(c) (0, 1) (d) (1, 0)

24. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having its centre (0, 3) is

(a) 4 (b) 3

(c) $\sqrt{12}$ (d) 7/2

25. The length of the latus rectum of an ellipse is one third of the major axis, its eccentricity would be

(a) 2/3 (b) $1/\sqrt{3}$

(c) $1/\sqrt{2}$ (d) $\sqrt{2/3}$

26. If the length of the major axis of an ellipse is three times the length of its minor axis, its eccentricity is

(a) 1/3 (b) $1/\sqrt{3}$

(c) $1/\sqrt{2}$ (d) $2/\sqrt{2}/3$

27. An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm, the necessary length of the string and the distance between the pins respectively in cms. are

(a) 6, $2\sqrt{5}$

(b) 6, $\sqrt{5}$

(c) 4, $2\sqrt{5}$

(d) None of these

28. The locus of mid-points of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$

(c) $x^2 + y^2 = a^2 + b^2$

(d) None of these

29. The locus of the point of intersection of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which meet at right angles, is

(a) a circle

(b) a parabola

(c) an ellipse

(d) a hyperbola

30. The number of real tangents that can be drawn to the ellipse $3x^2 + 5y^2 = 32$ passing through (3, 5) is

(a) 0 (b) 1

(c) 2 (d) 4

31. Equation to the ellipse whose centre is (–2, 3) and whose semi-axes are 3 and 2 and major axis is parallel to the x-axis, is given by

(a) $4x^2 + 9y^2 + 16x - 54y - 61 = 0$
(b) $4x^2 + 9y^2 - 16x + 54y + 61 = 0$
(c) $4x^2 + 9y^2 + 16x - 54y + 61 = 0$
(d) None of these

32. The foci of the ellipse $25(x+1)^2 + 9(y+2)^2 = 225$, are at
(a) (–1, 2) and (–1, –6)
(b) (–2, 1) and (–2, 6)
(c) (–1, –2) and (–2, –1)
(d) (–1, –2) and (–1, –6)

33. Tangents drawn from a point on the circle $x^2 + y^2 = 41$ to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ then tangents are at angle
(a) $\pi/4$ (b) $\pi/2$
(c) $\pi/3$ (d) $\pi/8$

34. Eccentricity of the ellipse, in which the angle between the straight lines joining the foci to an extremity of minor axis is $\pi/2$, is given by
(a) 1/2 (b) $1/\sqrt{2}$
(c) 1/3 (d) $1/\sqrt{3}$

35. If O is the centre, OA the semi-major axis and S the focus of an ellipse, the ecentric angle of any point P is
(a) $\angle POS$ (b) $\angle PSA$
(c) $\angle PAS$ (d) None of these

MULTIPLE CHOICE –II

Each question in this part has one or more than one correct answer(s). For each question write the letters a, b, c, d corresponding to the correct answer(s).

36. The locus of extremities of the latus rectum of the family of ellipse $b^2x^2 + y^2 = a^2b^2$ is
(a) $x^2 - ay = a^2$
(b) $x^2 - ay = b^2$
(c) $x^2 + ay = a^2$
(d) $x^2 + ay = b^2$

37. The distance of the point $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ from the centre is 2 if
(a) $\theta = \pi/2$
(b) $\theta = 3\pi/2$
(c) $\theta = 5\pi/2$
(d) $\theta = 7\pi/2$

38. The sum of the square of perpendiculars on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from two points on the minor axis, each at a distant ae from the centre, is
(a) $2a^2$ (b) $2b^2$
(c) $a^2 + b^2$ (d) $a^2 - b^2$

39. A latus rectum of an ellipse is a line
(a) passing through a focus
(b) through the centre
(c) perpendicular to the major axis
(d) parallel to the minor axis.

40. If latus rectum of the ellipse $x^2\tan^2\alpha + y^2\sec^2\alpha = 1$ is 1/2 then α $(0 < \alpha < \pi)$ is equal to
(a) $\pi/12$
(b) $\pi/6$
(c) $5\pi/12$
(d) None of these

41. In the ellipse
$$25x^2 + 9y^2 - 150x - 90y + 225 = 0$$
(a) foci are at (3, 1), (3, 9)
(b) $e = 4/5$
(c) centre is (5, 3)
(d) major axis is 6

42. Equation of tangent to the ellipse $x^2/9 + y^2/4 = 1$ which cut off equal intercepts on the axes is
(a) $y = x + \sqrt{(13)}$
(b) $y = -x + \sqrt{(13)}$
(c) $y = x - \sqrt{(13)}$
(d) $y = -x - \sqrt{(13)}$

43. The equation of tangent to the ellipse $x^2 + 3y^2 = 3$ which is perpendicular to the line $4y = x - 5$ is

(a) $4x+y+7=0$
(b) $4x+y-7=0$
(c) $4x+y+3=0$
(d) $4x+y-3=0$

44. If $P(\theta)$ and $Q\left(\frac{\pi}{2}+\theta\right)$ are two points one the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, locus of the mid-point of PQ is
(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$
(b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$
(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$
(d) None of these

45. An ellipse slides between two perpendicular straight lines. Then the locus of its centre is
(a) a parabola
(b) an ellipse
(c) a hyperbola
(d) a circle

46. If α, β are eccentric angles of the extremities of a focal chord of an ellipse, then eccentricity of the ellipse is
(a) $\frac{\cos\alpha+\cos\beta}{\cos(\alpha+\beta)}$
(b) $\frac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}$
(c) $\sec\alpha+\sec\beta$
(d) $\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$

47. The points where the normals to the ellipse $x^2+3y^2=37$ be parallel to the line :
(a) (5, 2) (b) (2, 5)
(c) (1, 3) (d) (–5, –2)

48. The length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ where mid-point is $\left(\frac{1}{2},\frac{2}{3}\right)$
(a) $\frac{1}{10}$ (b) $\frac{\sqrt{8161}}{10}$
(c) $\frac{\sqrt{8061}}{10}$ (d) None of these

49. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. the equation of the diameter conjugate to $ax-by=0$ is
(a) $bx+ay=0$
(b) $bx-ay=0$
(c) $a^3y+b^3x=0$
(d) $a^3y-b^3x=0$

50. The parametric representation of a point on the ellipse whose foci are (–1,0) and (7, 0) and eccentricity is 1/2, is
(a) $(3+8\cos\theta, 4\sqrt{3}\sin\theta)$
(b) $(8\cos\theta, 4\sqrt{3}\sin\theta)$
(c) $(3+4\sqrt{3}\cos\theta, 8\sin\theta)$
(d) None of these

Practice Test

M.M : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).
[10 × 2 = 20]

1. Let F_1, F_2 be two focii of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P. Then
(a) PN bisects $\angle F_1PF_2$
(b) PT bisects $\angle F_1PF_2$
(c) PT bisects angle $(180° - \angle F_1PF_2)$
(d) None of these

2. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. Let P and Q be the ponts (1, 2) and (2, 1) respectively. Then
(a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) P lies inside C but outside E

3. The tangent at a point $P(a\cos\phi, b\sin\phi)$ of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets its auxiliary circle in two points, the chord joining which subtends a right angle at the centre, then the eccentricity of the ellipse is

(a) $(1+\sin^2\phi)^{-1}$

(b) $(1+\sin^2\phi)^{-1/2}$

(c) $(1+\sin^2\phi)^{-3/2}$

(d) $(1+\sin^2\phi)^{-2}$

4. If (5, 12) and (24, 7) are the focii of a conic passing through the origin then the eccentricity of conic is

(a) $\frac{\sqrt{386}}{38}$ (b) $\frac{\sqrt{386}}{12}$

(c) $\frac{\sqrt{386}}{13}$ (d) $\frac{\sqrt{386}}{25}$

5. AB is a diameter of $x^2+9y^2=25$. The eccentric angle of A is $\pi/6$ then the eccentric angle of B is

(a) $5\pi/6$ (b) $-5\pi/6$

(c) $-2\pi/3$ (d) None of these

6. The eccentricity of the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of x and the straight line $\frac{x}{3}-\frac{y}{5}=1$ on the axis of y and whose axes lie along the axes of coordinates, is

(a) $\frac{3\sqrt{2}}{7}$ (b) $\frac{2\sqrt{3}}{7}$

(c) $\frac{\sqrt{3}}{7}$ (d) None of these

7. The eccentricity of an ellipse whose pair of a conjugate diameter are $y=x$ and $3y=-2x$ is

(a) 2/3 (b) 1/3

(c) $1/\sqrt{3}$ (d) None of these

8. The eccentric angles of the extremities of latus rectum to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by

(a) $\tan^{-1}\left(\pm\frac{ae}{b}\right)$

(b) $\tan^{-1}\left(\pm\frac{be}{a}\right)$

(c) $\tan^{-1}\left(\pm\frac{b}{ae}\right)$

(d) $\tan^{-1}\left(\pm\frac{a}{be}\right)$

9. A latus rectum of an ellipse is a line

(a) passing through a focus

(b) perpendicular to the major axis

(c) parallel to the minor axis

(d) through the centre

10. The tangents from which of the following points to the ellipse $5x^2+4y^2=20$ are perpendicular

(a) $(1, 2\sqrt{2})$ (b) $(2\sqrt{2}, 1)$

(c) $(2, \sqrt{5})$ (d) $(\sqrt{5}, 2)$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice–I

1. (b)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (a)
7. (c)	**8.** (b)	**9.** (c)	**10.** (c)	**11.** (a)	**12.** (d)
13. (b)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (d)	**18.** (c)

19. (a)	**20.** (b)	**21.** (a)	**22.** (d)	**23.** (b)	**24.** (a)
25. (d)	**26.** (d)	**27.** (a)	**28.** (a)	**29.** (a)	**30.** (c)
31. (c)	**32.** (a)	**33.** (b)	**34.** (b)	**35.** (d)	

Multiple Choice–II

36. (a), (c)	**37.** (a), (b), (c), (d)	**38.** (a)	**39.** (a), (c), (d)	**40.** (a), (c)
41. (a), (b)	**42.** (a), (b), (c), (d)	**43.** (a), (c)	**44.** (a)	**45.** (d)
46. (d)	**47.** (a, d) **48.** (d)	**49.** (c)	**50.** (a)	

Practice Test

1. (a), (c)	**2.** (d)	**3.** (b)	**4.** (a), (b)	**5.** (b)	**6.** (d)
7. (c)	**8.** (c)	**9.** (a), (b), (c)	**10.** (a), (b), (c), (d)		

26

HYPERBOLA

1. Standard Form of a Hyperbola :

The general form of standard Hyperbola is : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a & b are constants. (Fig. 1)

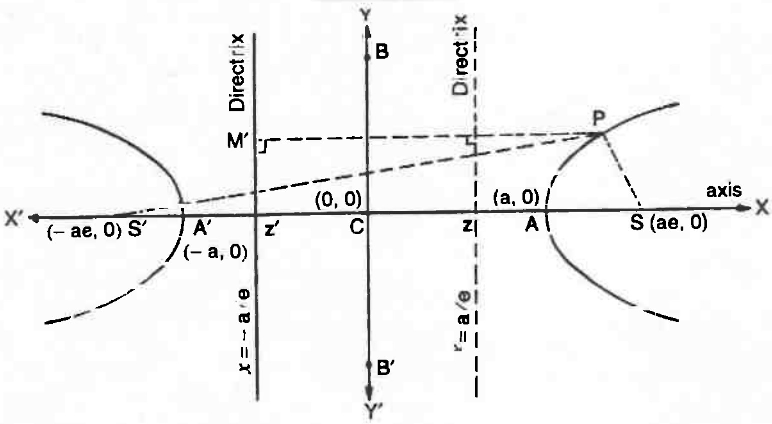


Fig. 1

2. Important Properties :

(i) $SP = e\,PM$ and $AS = e\,AZ$

(ii) Co-ordinate of centre $C\,(0, 0)$.

(iii) $AA' = 2a$ is the **transverse axis** of the Hyperbola.

(iv) $BB' = 2b$ is the **conjugate axis** of the Hyperbola.

(v) Co-ordinates of **vertices** A and A' are $(\pm a, 0)$ & **extremities** of **minor axis** B and B' are $(0, \pm b)$

(vi) Relation in a, b & e is $b^2 = a^2(e^2 - 1)$

(vii) Co-ordinates of the **foci** S and S' are $(\pm ae, 0)$

(viii) Co-ordinates of the **feet of directrices** are $\left(\pm \frac{a}{e}, 0\right)$

(ix) Equation of **directrix** $x = + a/e$

(x) Equation of **latus rectum** $x = \pm ae$ and length $LL' = L_1L_1' = \frac{2b^2}{a}$

(xi) **Ends of** the latus rectum, are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$, $L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(xii) **Focal radii :** $SP = ex - a$ and $S'P = ex + a$

$\Rightarrow$ $S'P - SP = 2a$ = Transverse axis.

3. General Equation of Hyperbola :

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix, let $P(x, y)$ be any point on the hyperbola, (Fig. 2) then by definition.

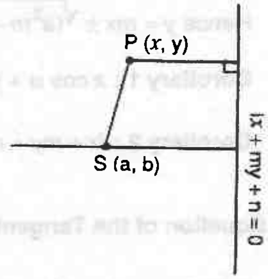


Fig. 2

$\Rightarrow$ $$SP = e\,PM \quad (e > 1)$$

$\Rightarrow$ $$(x-a)^2 + (y-b)^2 = \frac{e^2 (lx+my+n)^2}{(l^2+m^2)}$$

$\Rightarrow$ $$(l^2+m^2)\{(x-a)^2+(y-b)^2\} = e^2 (lx+my+n)^2.$$

4. Parametric Equation of the hyperbola :

The parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $x = a\sec\phi$, $y = b\tan\phi$, where ϕ is the parameter. Since the point $(a\sec\phi, b\tan\phi)$ satisfies the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, therefore the parametric co-ordinates of any point on the hyperbola is $(a\sec\phi, b\tan\phi)$ also the point $(a\sec\phi, b\tan\phi)$ is reffered as ϕ point on the hyperbola. $\phi \in [0, 2\pi)$

5. Auxiliary circle :

The circle described on transverse axis of the hyperbola as diameter is called auxiliary circle and so its equation is

$$x^2 + y^2 = a^2$$

Let P be any point on the hyperbola. Draw perpendicular PN to x-axis. The tangent from N to the auxiliary circle touches at Q, P and Q are called corresponding points on hyperbola and auxiliary circle and ϕ is the eccentric angle of the point P on the hyperbola.

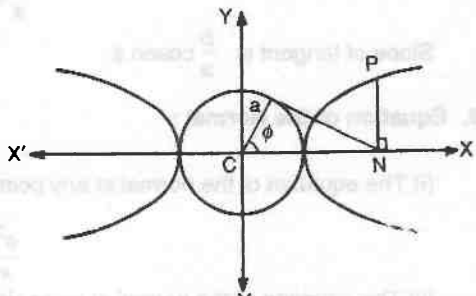


Fig. 3

6. Point and Hyperbola :

The point $P(x_1, y_1)$ lies outside, on or inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0, = 0, < 0.$$

7. Hyperbola and a Line :

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the given line be $y = mx + c$

Solving the line and hyperbola we get

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

i.e. $$(a^2m^2 - b^2)x^2 + 2mca^2x + a^2(c^2+b^2) = 0.$$

Above equation being a quadratic in x.

$\therefore$ $$\text{discriminant} = b^2\{(a^2m^2 - b^2) - c^2\}$$

Hence the line intersects the hyperbola is 2 distinct points if $a^2m^2 - b^2 > c^2$, in one point if $c^2 = a^2m^2 - b^2$ and does not intersect if $a^2m^2 - b^2 < c^2$.

$\therefore$ $y = mx \pm \sqrt{(a^2m^2 - b^2)}$ touches the hyperbola and condition for tangency

$$c^2 = a^2m^2 - b^2.$$

Hence $y = mx \pm \sqrt{(a^2m - b^2)}$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $\left(\pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Corollary 1 : $x \cos \alpha + y \sin \alpha = p$ is a tangent if

$$p^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha$$

Corollary 2 : $lx + my + n = 0$ is a tangent if

$$n^2 = a^2 l^2 - b^2 m^2$$

8. Equation of the Tangent :

(i) The equation of the tangent at any point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Slope of tangent is $\frac{b^2 x_1}{a^2 y_1}$ (Note)

(ii) the equation of tangent at any point 'ϕ' is

$$\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1$$

Slope of tangent is $\frac{b}{a} \operatorname{cosec} \phi$.

9. Equation of the Normal :

(i) The equation of the normal at any point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

(ii) The equation of the normal at any point 'ϕ' is

$$ax \cos \phi + by \cot \phi = a^2 + b^2.$$

10. Equation of Chord with Mid point (x_1, y_1) :

The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose mid point be (x_1, y_1) is

$$T = S_1$$

where

$$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

$$S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = 0$$

11. Chord of Contact :

If PA and PB be the tangents through point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equation of the chord of contact AB is (Fig. 4).

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \text{ or } T = 0 \text{ (at } x_1, y_1)$$

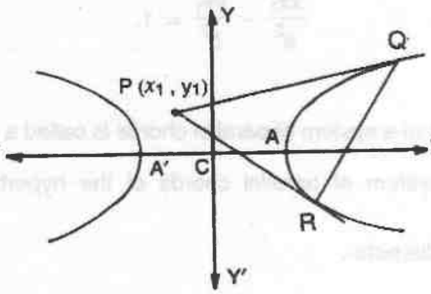


Fig. 4

12. Pair of Tangents :

If $P(x_1, y_1)$ be any point out side the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and a pair of tangents PA, PB can be drawn to it from P. (Fig. 5).

then the equation of pair of tangents of PA & PB is

$$SS_1 = T^2$$

where
$$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

$$S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = 0$$

$$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

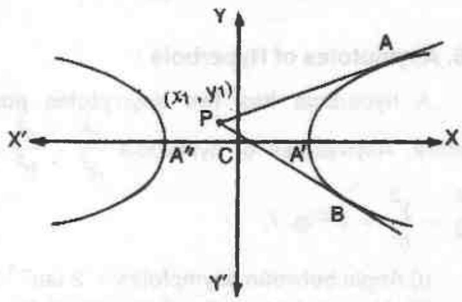


Fig. 5

13. Pole and Polar :

Let $P(x_1, y_1)$ be any point inside or outside the hyperbola. Draw chords AB and $A'B'$ passing through P.

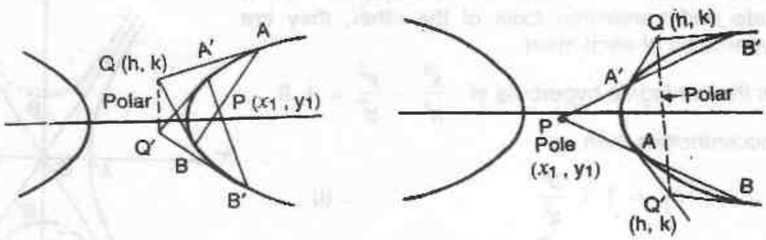


Fig. 6

If tangents to the hyperbola at A & B meet at $Q(h, k)$, then locus of Q is called polar of P w.r.t. hyperbola and P is called the pole and if tangents to the hyperbola at A' & B' meet at Q', then the straight line QQ' is polar with P its pole.

Hence equation of polar of $P(x_1, y_1)$ with respect to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

14. Diameter :

The locus of the middle points of a system of parallel chords is called a diameter.

If $y = mx + c$ represent a system of parallel chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then the line $y = \frac{b^2}{a^2 m} x$ is the equation of the diameter.

15. Conjugate Diameters :

Two diameters are said to be conjugate when each bisects all chords parallel to the others.
If $y = mx$ and $y = m_1 x$ be two conjugate diameters of a hyperbola then

$$mm_1 = \frac{b^2}{a^2}$$

Property of conjugate diameters : If a pair of conjugate diameters of any hyperbola be given, only one of them will intersect it in real points.

16. Asymptotes of Hyperbola :

A hyperbola has two asymptotes passing through its centre. Asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ Fig. 7.

(i) Angle between asymptotes $= 2 \tan^{-1} (b/a)$

(ii) Asymptotes are the diagonals of the rectangle passing through A, B, A', B' with sides parallel to axes.

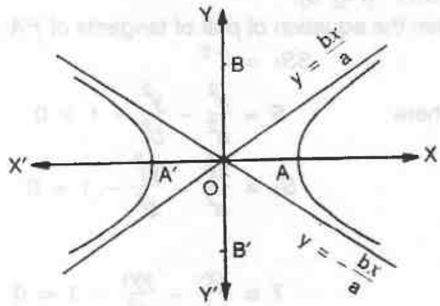


Fig. 7

17. Conjugate Hyperbola :

If two hyperbolas be such that transverse and conjugate axes of one be the conjugate and transverse axes of the other, they are called **conjugate** hyperbolas of each other.

$\therefore$ $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is the conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If e_1 and e_2 are their eccentricities then

$$e_1^2 = 1 + \frac{b^2}{a^2} \quad \text{...(i)}$$

and

$$e_2^2 = 1 + \frac{a^2}{b^2} \quad \text{...(ii)}$$

From (i) & (ii) we get

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

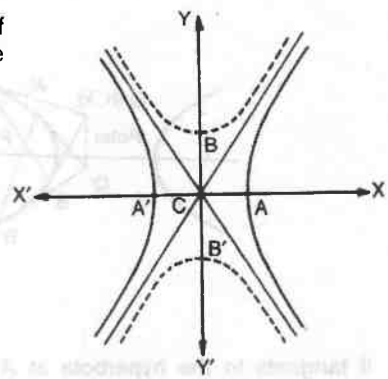


Fig. 8

Note : Complete line is Hyperbola & Dotted line is conjugate hyperbola.

18. Rectangular or Equilateral Hyperbola :

If the lengths of transverse and conjugate axes of any hyperbola be equal, it is called **rectangular or equilateral hyperbola.**

OR

If asymptotes of the standard hyperbola are perpendicular to each other then it is known as **rectangular hyperbola.**

According to the first definition :

Thus when $a = b, \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ becomes $x^2 - y^2 = a^2$

Eccentricity, $e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{2}$

According to the second definition :

$$2\tan^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{2}$$

$\Rightarrow$ $$\tan^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{4}$$

$\Rightarrow$ $$\frac{b}{a} = 1$$

$\Rightarrow$ $$a = b$$

then $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ becomes $x^2 - y^2 = a^2$.

is the general form of the equation of the rectangular hyperbola.

19. The Rectangular hyperbola $xy = c^2$:

Its asymptotes coincide with the co-ordinate axes, then its equation becomes $xy - c^2$.

Parametric Equations and t-point :

Since $x = ct, y = \dfrac{c}{t}$ satisfy $xy = c^2$, $(x, y) = \left(ct, \dfrac{c}{t}\right)$ is called a 't' point with parameter t.

Properties :

(i) Equation of the chord joining t_1 & t_2 is

$$x + y t_1 t_2 - c(t_1 + t_2) = 0$$

(ii) Equation of tangent at 't' is

$$x + yt^2 - 2ct = 0$$

(iii) Equation of normal at 't' is

$$xt^3 - yt - ct^4 + c = 0$$

(iv) Equation of tangent at (x_1, y_1) is

$$xy_1 + yx_1 = 2c^2$$

(v) Equation of normal at (x_1, y_1) is

$$xx_1 - yy_1 = x_1^2 - y_1^2$$

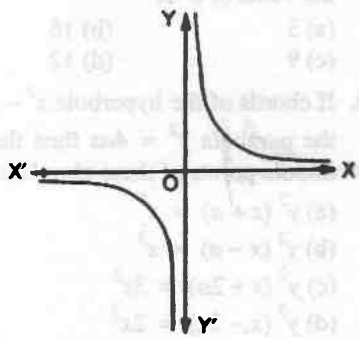


Fig. 9

21. Director Circle :

The locus of the point of intersection of the tangents to a hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ which are perpendicular to each other is called **Director circle.**

Hence the equation of director circle of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = a^2 - b^2$$

22. Reflection Property of a Hyperbola :

If an incoming light ray passing through one focus (S) strike convex side of the hyperbola then it will get reflected towards other focus (S'). fig. 10.

$$\therefore \qquad \angle TPS' = \angle LPM = \alpha .$$

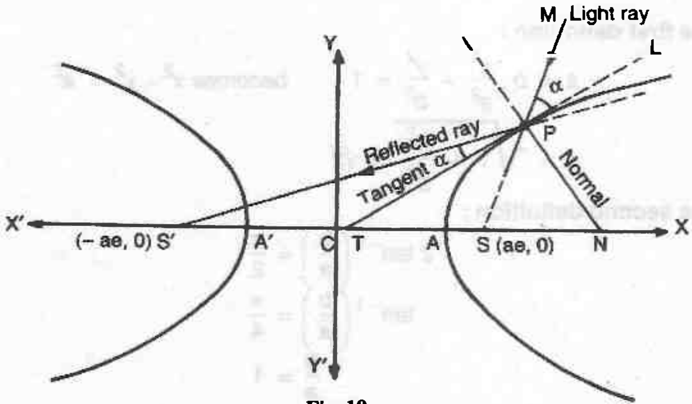


Fig. 10

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is
(a) 3 (b) 16
(c) 9 (d) 12

2. If chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$ then the locus of the middle points of these chords is the curve
(a) $y^2(x+a) = x^3$
(b) $y^2(x-a) = x^3$
(c) $y^2(x+2a) = 3x^3$
(d) $y^2(x-2a) = 2x^3$

3. If the sum of the slopes of the normals from a point P on hyperbola $xy = c^2$ is constant k $(k > 0)$, then the locus of P is
(a) $y^2 = k^2c$ (b) $x^2 = kc^2$
(c) $y^2 = ck^2$ (d) $x^2 = ck^2$

4. If $(a-2)x^2 + ay^2 = 4$ represents a rectangular hyperbola then a equals
(a) 0 (b) 2
(c) 1 (d) 3

5. The number of point(s) outside the hyperbola $\frac{x^2}{25} - \frac{y^2}{36} = 1$ from where two perpendicular tangents can be drawn to the hyperbola is/are
(a) 1 (b) 2
(c) infinite (d) zero

6. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then the eccentricity e of the hyperbola satisfies
(a) $1 < e < \frac{2}{\sqrt{3}}$ (b) $e = \frac{2}{\sqrt{3}}$
(c) $e = \sqrt{3}/2$ (d) $e > \frac{2}{\sqrt{3}}$

7. The equations of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$ are
(a) $2x^2 + 5xy + 2y^2 - 11x - 7y - 5 = 0$
(b) $2x^2 + 4xy + 2y^2 - 7x - 11y + 5 = 0$
(c) $2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$
(d) None the these

8. The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M respectively. If locus of the mid point of LM is hyperbola, then eccentricity of the hyperbola is

(a) $\frac{e+1}{e-1}$ (b) $\frac{e}{\sqrt{e^2-1}}$

(d) e (d) None of these

9. Consider the set of hyperbola $xy = k, k \in R$. Let e_1 be the eccentricity when $k = 4$ and e_2 be the eccentricity when $k = 9$ then $e_1 - e_2 =$

(a) –1 (b) 0

(c) 2 (d) 3

10. The eccentaicity of the hyperbola whose asymptotes are $3x + 4y = 2$ and $4x - 3y + 5 = 0$ is

(a) 1 (b) 2

(c) $\sqrt{2}$ (d) None of these

11. If a variable straight line $x \cos\alpha + y \sin\alpha = p$, which is a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(b > a)$, subtend a right angle at the centre of the hyperbola then it always touches a fixed circle whose radius is

(a) $\frac{ab}{\sqrt{b-2a}}$ (b) $\frac{a}{\sqrt{a-b}}$

(c) $\frac{ab}{\sqrt{b^2-a^2}}$ (d) $\frac{ab}{b\sqrt{(b+a)}}$

12. An ellipse has eccentiricity 1/2 and one focus at the point $P(1/2, 1)$. Its one directrix is the common tangent nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse in standard form is

(a) $9x^2 + 12y^2 = 108$

(b) $9(x - 1/3)^2 + 12(y - 1)^2 = 1$

(c) $9(x - 1/3)^2 + 4(y - 1)^2 = 36$

(d) None of these

13. The equation of the line passing through the centre of a rectangular hyperbola is $x - y - 1 = 0$. If one of its asymptote is $3x - 4y - 6 = 0$, the equation of the other asymptote is

(a) $4x - 3y + 8 = 0$ (b) $4x + 3y + 17 = 0$

(c) $3x - 2y + 15 = 0$ (d) None of these

14. The condition that a straight line with slope m will be normal to parabola $y^2 = 4ax$ as well as a tangent to rectangular hyperbola $x^2 - y^2 = a^2$ is

(a) $m^6 - 4m^2 + 2m - 1 = 0$

(b) $m^4 + 3m^2 + 2m + 1 = 0$

(c) $m^6 - 2m = 0$

(d) $m^6 + 4m^4 + 3m^2 + 1 = 0$

15. If e is the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ is angle between the asymptotes, then $\cos \theta/2 =$

(a) $\frac{1-e}{e}$ (b) $\frac{1}{e} - 1$

(c) $1/e$ (d) None of these

16. If $H(x, y) = 0$ represent the equation of a hyperbola and $A(x, y) = 0$, $C(x, y) = 0$ the equations of its asymptotes and the conjugate hyperbola respectively then for any point (α, β) in the plane; $H(\alpha, \beta)$, $A(\alpha, \beta)$ and $C(\alpha, \beta)$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

17. The eccentricity of the conic $4(2y - x - 3)^2 - 9(2x + y - 1)^2 = 80$ is

(a) 2 (b) 1/2

(c) $\sqrt{13/3}$ (d) 2.5

18. If e and e' be the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e'^2} =$

(a) 0 (b) 1

(c) 2 (d) None of these

19. The line $x \cos\alpha + y \sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if

(a) $a^2 \cos^2\alpha - b^2 \sin^2\alpha = p^2$

(b) $a^2 \cos^2\alpha - b^2 \sin^2\alpha = p$

(c) $a^2 \cos^2\alpha + b^2 \sin^2\alpha = p^2$

(d) $a^2 \cos^2\alpha + b^2 \sin^2\alpha = p$.

20. The diameter of $16x^2 - 9y^2 = 144$, which is conjugate to $x = 2y$, is

(a) $y = \frac{16x}{9}$ (b) $y = \frac{32x}{9}$

(c) $x=\frac{16y}{9}$ (d) $x=\frac{32y}{9}$

21. $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\tan\theta/2\tan\phi/2$ equals to

(a) $\frac{e-1}{e+1}$ (b) $\frac{1-e}{1+e}$

(c) $\frac{1+e}{1-e}$ (d) $\frac{e+1}{e-1}$

22. The equation of the hyperbola whose foci are (6, 5), (– 4, 5) and ecountricity 5/4 is

(a) $\frac{(x-1)^2}{16}-\frac{(y-5)^2}{9}=1$

(b) $\frac{x^2}{16}-\frac{y^2}{9}=1$

(c) $\frac{(x-1)^2}{16}-\frac{(y-5)^2}{9}=1$

(d) None of these

23. The locus of the middle points of chords of hyperbola $3x^2-2y^2+4x-6y=0$ parallel to $y=2x$ is

(a) $3x-4y=4$

(b) $3y-4x+4=0$

(c) $4x-4y=3$

(d) $3x-4y=2$

24. Area of the triangle formed by the lines $x-y=0$, $x+y=0$ and any tangent to the hyperbola $x^2-y^2=a^2$ is

(a) $|a|$ (b) $\frac{1}{2}|a|$

(c) a^2 (d) $\frac{1}{2}a^2$

25. A rectangular hyperbola whose centre is C is cut by any circle of radius r in four points P, Q, R and S. Then $CP^2+CQ^2+CR^2+CS^2=$

(a) r^2 (b) $2r^2$

(c) $3r^2$ (d) $4r^2$

MULTIPLE CHOICE –II

Each question in this part has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

26. The equation $16x^2-3y^2-32x-12y-44=0$ represents a hyperbola with

(a) length of the transverse axis = $2\sqrt{3}$

(b) length of the conjugate axis = 8

(c) Centre at (1, –2)

(d) eccentricity = $\sqrt{19}$

27. The equation of a tangent to the hyperbola $3x^2-y^2=3$, parallel to the line $y=2x+4$ is

(a) $y=2x+3$ (b) $y=2x+1$

(c) $y=2x-1$ (d) $y=2x+2$

28. Equation of a tangent passing through (2, 8) to the hyperbola $5x^2-y^2=5$ is

(a) $3x-y+2=0$ (b) $3x+y+14=0$

(c) $23x-3y-22=0$ (d) $3x-23y+178$

29. If the line $ax+by+c=0$ is a normal to the hyperbola $xy=1$, then

(a) $a>0, b>0$ (b) $a>0, b<0$

(c) $a<0, b>0$ (d) $a<0, b<0$

30. If m_1 and m_2 are the slopes of the tangents to the hyperbola $x^2/25-y^2/16=1$ which pass through the point (6, 2) then

(a) $m_1+m_2=24/11$

(b) $m_1m_2=20/11$

(c) $m_1+m_2=48/11$

(d) $m_1m_2=11/20$

31. Product of the lengths of the perpendiculars drawn from foci on any tangent to the hyperbola $x^2/a^2-y^2/b^2=1$ is

(a) $\frac{1}{2}b^2$ (b) b^2

(c) a^2 (d) $\frac{1}{2}a^2$

32. The locus of the point of intersection of two perpendicular tangents to the hyperbola $x^2/a^2-y^2/b^2=1$ is

(a) director circle (b) $x^2+y^2=a^2$

(c) $x^2+y^2=a^2-b^2$ (d) $x^2+y^2=a^2+b^2$

33. The locus of the point of intersection of the line $\sqrt{3}x-y-4\sqrt{3}k=0$ and $\sqrt{3}kx+ky-4\sqrt{3}=0$ is a hyperbola of eccentricity

(a) 1 (b) 2

(c) 2·5 (d) $\sqrt{3}$

34. If a triangle is inscribed in a rectangular hyperbola, its orthocentre lies
(a) inside the curve (b) outside the curve
(c) on the curve (d) None of these

35. Equation of the hyperbola passing through the point (1, −1) and having asymptotes $x+2y+3=0$ and $3x+4y+5=0$ is
(a) $3x^2-10xy+8y^2-14x+22y+7=0$
(b) $3x^2+10xy+8y^2-14x+22y+7=0$
(c) $3x^2-10xy-8y^2+14x+22y+7=0$
(d) $3x^2+10xy+8y^2+14x+22y+7=0$

36. The equation of tangent parallel to $y=x$ drawn to $\frac{x^2}{3}-\frac{y^2}{2}=1$ is
(a) $x-y+1=0$ (b) $x-y-2=0$
(c) $x+y-1=0$ (d) $x-y-1=0$

37. The normal to the rectangular hyperbola $xy=c^2$ at the point 't_1' meets the curve again at the point 't_2'. Then the value of $t_1^3 t_2$ is
(a) 1 (b) −1
(c) c (d) $-c$

38. If $x=9$ is the chord of contact of the hyperbola $x^2-y^2=9$, then the equation of the corresponding pair of tangents is
(a) $9x^2-8y^2+18x-9=0$
(b) $9x^2-8y^2-18x+9=0$
(c) $9x^2-8y^2-18x-9=0$
(d) $9x^2-8y^2+18x+9=0$

39. Tangents drawn from a point on the circle $x^2+y^2=9$ to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$, then tangents are at angle
(a) $\pi/4$ (b) $\pi/2$
(c) $\pi/3$ (d) $2\pi/3$

40. If e and e_1 are the eccentricities of the hyperbola $xy=c^2$ and $x^2-y^2=c^2$, then $e^2+e_1^2=$
(a) 1 (b) 4
(c) 6 (d) 8

Practice Test

M.M : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).
[10 × 2 = 20]

1. The points of intersection of the curves whose parametric equations are $x=t^2+1$, $y=2t$ and $x=2s$, $y=\frac{2}{s}$ is given by
(a) (1, −3) (b) (2, 2)
(c) (−2, 4) (d) (1, 2)

2. The equations to the common tangents to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are
(a) $y=\pm x+\sqrt{(b^2-a^2)}$
(b) $y=\pm x+\sqrt{(a^2-b^2)}$
(c) $y=\pm x\pm(a^2-b^2)$
(d) $y=\pm x\pm\sqrt{(a^2+b^2)}$

3. The asymptotes of the hyperbola $xy=hx+ky$ are
(a) $x=k,\ y=h$ (b) $x=h,\ y=k$
(c) $x=h,\ y=h$ (d) $x=k,\ y=k$

4. If $P(x_1,y_1)$, $Q(x_2,y_2)$, $R(x_3,y_3)$ and $S(x_4,y_4)$ and 4 concyclic points on the rectangular hyperbola $xy=c^2$, the co-ordinates of the orthocentre of the ΔPQR are
(a) $(x_4,-y_4)$ (b) (x_4,y_4)
(c) $(-x_4,-y_4)$ (d) $(-x_4,y_4)$

5. The equation of a hyperbola, conjugate to the hyperbola $x^2+3xy+2y^2+2x+3y=0$ is
(a) $x^2+3xy+2y^2+2x+3y+1=0$
(b) $x^2+3xy+2y^2+2x+3y+2=0$
(c) $x^2+3xy+2y^2+2x+3y+3=0$
(d) $x^2+3xy+2y^2+2x+3y+4=0$

6. If the tangent and normal to a rectangular hyperbola cut off intercepts x_1 and x_2 on one axis and y_1 and y_2 on the other axis, then

(a) $x_1 y_1 + x_2 y_2 = 0$
(b) $x_1 y_2 + x_2 y_1 = 0$
(c) $x_1 x_2 + y_1 y_2 = 0$
(d) None of these

7. A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse and conjugate axes in M and N and the lines MP and NP are drawn at right angles to the axes. The locus of P is

(a) The parabola $y^2 = 4a(x + b)$
(b) The circle $x^2 + y^2 = ab$
(c) The ellipse $b^2x^2 + a^2y^2 = a^2 + b^2$
(d) The hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$

8. The line $y = x + 5$ touches

(a) the parabola $y^2 = 20x$
(b) the ellipse $9x^2 + 16y^2 = 144$
(c) the hyperbola $\frac{x^2}{29} - \frac{y^2}{4} = 1$
(d) the circle $x^2 + y^2 = 25$

9. A ray emanating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8, then the equation of the reflected ray after first reflection is (P lies in first quadrant)

(a) $\sqrt{3}x - y + 7 = 0$
(b) $3\sqrt{3}x - 13y + 15\sqrt{3} = 0$
(c) $3\sqrt{3}x + 13y - 15\sqrt{3} = 0$
(d) $\sqrt{3}x + y - 14 = 0$

10. A straight line touches the rectangular hyperbola $9x^2 - 9y^2 = 8$ and the parabola $y^2 = 32x$. The equation of the line is

(a) $9x + 3y - 8 = 0$ (b) $9x - 3y + 8 = 0$
(c) $9x + 3y + 8 = 0$ (d) $9x - 3y - 8 = 0$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (c) **2.** (b) **3.** (b) **4.** (c) **5.** (d) **6.** (d)
7. (c) **8.** (b) **9.** (b) **10.** (c) **11.** (c) **12.** (b)
13. (b) **14.** (d) **15.** (c) **16.** (a) **17.** (c) **18.** (b)
19. (a) **20.** (b) **21.** (b) **22.** (a) **23.** (a) **24.** (a)
25. (d)

Multiple Choice –II

26. (a), (b, (c) **27.** (b), (c) **28.** (a), (c) **29.** (b), (c) **30.** (b), (c) **31.** (b)
32. (a), (c) **33.** (b) **34.** (c) **35.** (b) **36.** (a), (d) **37.** (b)
38. (b) **39.** (b) **40.** (b)

Practice Test

1. (b) **2.** (b) **3.** (a) **4.** (b), (c) **5.** (b) **6.** (c)
7. (d) **8.** (a), (b), (c) **9.** (b) **10.** (b), (c)

TRIGONOMETRY

27

TRIGONOMETRICAL, RATIOS AND IDENTITIES

§ 27.1. Some Important Results

(i) $\cos n\pi = (-1)^n, \sin n\pi = 0$ If $n \in I$

(ii) $\cos \frac{n\pi}{2} = 0, \sin \frac{n\pi}{2} = (-1)^{(n-1)/2}$, If n is odd integer.

(iii) $\cos(n\pi + \theta) = (-1)^n \cos\theta$, If $n \in I$

and $\sin(n\pi + \theta) = (-1)^n \sin\theta$, If $n \in I$

(iv) $\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{(n+1)/2} \sin\theta$, If n is odd integer.

and $\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{(n-1)/2} \cos\theta$, If n is odd integer.

§ 27.2. For any three Angles *A*, *B*, *C*.

(i) $\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B + \sin A \sin B \sin C.$

(ii) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin C \sin A - \cos C \sin A \sin B.$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(iv) $\cot(A+B+C) = \dfrac{\cot A \cot B \cot C - (\cot A + \cot B + \cot C)}{\cot A \cot B + \cot B \cot C + \cot C \cot A - 1}.$

§ 27.3. Greatest and least value of $(a\cos\theta + b\sin\theta)$

i.e., $-\sqrt{(a^2+b^2)} \le (a\cos\theta + b\sin\theta) \le \sqrt{(a^2+b^2)}$

§ 27.4. Some Important Identities : If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iii) $\sin A + \sin B + \sin C = 4 \cos A/2 \cos B/2 \cos C/2.$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin A/2 \sin B/2 \sin C/2$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(vii) $\tan A/2 \tan B/2 + \tan B/2 \tan C/2 + \tan C/2 \tan A/2 = 1$

(viii) $\cot A/2 + \cot B/2 + \cot C/2 = \cot A/2 \cot B/2 \cot C/2.$

(ix) $\sin 2mA + \sin 2mB + \sin 2mC = (-1)^{m+1} . 4 \sin mA \sin mB \sin mC.$

(x) $\cos mA + \cos mB + \cos mC = 1 \pm 4 \sin \frac{mA}{2} \sin \frac{mB}{2} \sin \frac{mC}{2}$

according as m is of the form $4n+1$ or $4n+3$

Trigonometric Rations 0° – 90°

	0°	7.5°	15°	18°	22.5°	30°	36°	45°	60°	67.5°	75°	90°
sin	0	$\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2}-2}{2}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	1
cos	1	$\frac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	0
tan	0	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{3}$	$\sqrt{2}+1$	$2+\sqrt{3}$	∞
cot	∞	$(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$	$(\sqrt{6}-\sqrt{2})$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{3}$	$\sqrt{\left(2-\frac{2}{\sqrt{5}}\right)}$	1	$\frac{1}{\sqrt{3}}$	$\sqrt{2}-1$	$2-\sqrt{3}$	0
sec	1	$\sqrt{(16-10\sqrt{2}+8\sqrt{3}-6\sqrt{6})}$	$(\sqrt{6}-\sqrt{2})$	$\sqrt{\left(2-\frac{2}{\sqrt{5}}\right)}$	$\sqrt{4-2\sqrt{2}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{5}-1$	$\sqrt{2}$	2	$\sqrt{4+2\sqrt{2}}$	$\sqrt{6}+\sqrt{2}$	∞
cosec	∞	$\sqrt{(16+10\sqrt{2}+8\sqrt{3}+6\sqrt{6})}$	$(\sqrt{6}+\sqrt{2})$	$\sqrt{5}+1$	$\sqrt{4+2\sqrt{2}}$	2	$\sqrt{\left(2-\frac{2}{\sqrt{5}}\right)}$	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	$\sqrt{4+2\sqrt{2}}$	$\sqrt{6}+\sqrt{2}$	1

Two very useful identities :

(xi) $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right)$

(xii) $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\gamma+\alpha}{2}\right)$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ then the value of $x^2+y^2+z^2$ is independent of
 (a) θ, ϕ (b) r, θ
 (c) r, ϕ (d) r

2. If $0° < \theta < 180°$ then $\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2(1+\cos\theta)}}}}$ then being n number of 2's, is equal to
 (a) $2\cos\left(\frac{\theta}{2^n}\right)$ (b) $2\cos\left(\frac{\theta}{2^{n-1}}\right)$
 (c) $2\cos\left(\frac{\theta}{2^{n+1}}\right)$ (d) None of these

3. If $\tan\alpha/2$ and $\tan\beta/2$ are the roots of the equation $8x^2 - 26x + 15 = 0$ then $\cos(\alpha+\beta)$ is equal to
 (a) $-\frac{627}{725}$ (b) $\frac{627}{725}$
 (c) -1 (d) None of these

4. If $a\sec\alpha - c\tan\alpha = d$ and $b\sec\alpha + d\tan\alpha = c$ then
 (a) $a^2+c^2 = b^2+d^2$ (b) $a^2+d^2 = b^2+c^2$
 (c) $a^2+b^2 = c^2+d^2$ (d) $ab = cd$

5. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r\theta$ for all real θ then
 (a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$
 (c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - 3n - 3$

6. If θ is an acute angle and $\tan\theta = \frac{1}{\sqrt{7}}$, then the value of $\frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta}$ is
 (a) 3/4 (b) 1/2
 (c) 2 (d) 5/4

7. $\tan 7\frac{1}{2} =$
 (a) $\frac{2\sqrt{2}-(1+\sqrt{3})}{\sqrt{3}-1}$ (b) $\frac{1+\sqrt{3}}{1-\sqrt{3}}$
 (c) $\frac{1}{\sqrt{3}}+\sqrt{3}$ (d) $2\sqrt{2}+\sqrt{3}$

8. The maximum value of $\sin(x+\pi/6)+\cos(x+\pi/6)$ in the interval $(0, \pi/2)$ is alttained at
 (a) $\pi/12$ (b) $\pi/6$
 (c) $\pi/3$ (d) $\pi/2$

9. The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$, where α, β, γ are real numbers satisfying $\alpha+\beta+\gamma = \pi$ is
 (a) +ve (b) –ve
 (c) zero (d) –3

10. If $\sin\alpha = \sin\beta$ and $\cos\alpha = \cos\beta$, then
 (a) $\sin\left(\frac{\alpha+\beta}{2}\right) = 0$ (b) $\cos\left(\frac{\alpha+\beta}{2}\right) = 0$
 (c) $\sin\left(\frac{\alpha-\beta}{2}\right) = 0$ (d) $\cos\left(\frac{\alpha-\beta}{2}\right) = 0$

11. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is equal to
 (a) 1/2 (b) $\cos\pi/8$
 (c) 1/8 (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

12. If $A + C = B$, then $\tan A\tan B\tan C =$
 (a) $\tan A + \tan B + \tan C$
 (b) $\tan B - \tan C - \tan A$

(c) $\tan A + \tan C - \tan B$
(d) $-(\tan A \tan B + \tan C)$

13. If A lies in the third quadrant and $3 \tan A - 4 = 0$, then

$$5 \sin 2A + 3 \sin A + 4 \cos A =$$

(a) 0 (b) $-\frac{24}{5}$
(c) $\frac{24}{5}$ (d) $\frac{48}{5}$

14. If $\sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 3$, then $\cos \theta_1 + \cos \theta_2 + \cos \theta_3 =$
(a) 3 (b) 2
(c) 1 (d) 0

15. If $\sin x + \sin^2 x = 1$, then $\cos^8 x + 2 \cos^6 x + \cos^4 x =$
(a) -1 (b) 0
(c) 1 (d) 2

16. If $x = y \cos \frac{2\pi}{3} = z \cos \frac{4\pi}{3}$, then $xy + yz + zx =$
(a) -1 (b) 0
(c) 1 (d) 2

17. If $a \sin^2 x + b \cos^2 x = c$, $b \sin^2 y + a \cos^2 y = d$ and $a \tan x = b \tan y$, then $\frac{a^2}{b^2}$ is equal to
(a) $\frac{(b-c)(d-b)}{(a-d)(d-b)}$ (b) $\frac{(a-d)(c-a)}{(b-c)(d-b)}$
(c) $\frac{(d-a)(c-a)}{(b-c)(d-b)}$ (d) $\frac{(b-c)(b-d)}{(a-c)(a-d)}$

18. If $0 < \alpha < \pi/6$ and $\sin \alpha + \cos \alpha = \sqrt{7}/2$, then $\tan \alpha/2 =$
(a) $\frac{\sqrt{7}-2}{3}$ (b) $\frac{\sqrt{7}+2}{3}$
(c) $\frac{\sqrt{7}}{3}$ (d) None of these

19. The value of $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}$ is
(a) 1 (b) -1
(c) 1/2 (d) $-1/2$

20. If $\pi < \alpha < \frac{3\pi}{2}$, then the expression $\sqrt{4 \sin^4 \alpha + \sin^2 2\alpha} + 4 \cos^2 \left(\frac{\pi}{4} - \frac{\alpha}{2} \right)$ is equal to
(a) $2 + 4 \sin \alpha$ (b) $2 - 4 \sin \alpha$
(c) 2 (d) None of these

21. The value of $\cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{3\pi}{15} \cos \frac{4\pi}{15} \cos \frac{5\pi}{15} \cos \frac{6\pi}{15} \cos \frac{7\pi}{15}$ is
(a) $\frac{1}{2^6}$ (b) $\frac{1}{2^7}$
(c) $\frac{1}{2^8}$ (d) None of these

22. If $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2} \right)$, then the value of $\frac{\sin (\alpha + \beta + \gamma)}{\sin \alpha + \sin \beta + \sin \gamma}$ is
(a) < 1 (b) > 1
(c) $= 1$ (d) None of these

23. If $\frac{x}{\cos \alpha} = \frac{y}{\cos \left(\alpha - \frac{2\pi}{3} \right)} = \frac{z}{\cos \left(\alpha + \frac{2\pi}{3} \right)}$, then $x + y + z =$
(a) 1 (b) 0
(c) -1 (d) None of these

24. If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C =$
(a) $1 - 4 \cos A \cos B \cos C$
(b) $4 \sin A \sin B \sin C$
(c) $1 + 2 \cos A \cos B \cos C$
(d) $1 - 4 \sin A \sin B \sin C$

25. If $\sum_{i=1}^{n} \cos \theta_i = n$, then $\sum_{i=1}^{n} \sin \theta_i =$
(a) $n - 1$ (b) 0
(c) n (d) $n + 1$

26. If $\cos \alpha + \cos \beta = 0 = \sin \alpha + \sin \beta$, then $\cos 2\alpha + \cos 2\beta =$
(a) $-2 \sin (\alpha + \beta)$ (b) $-2 \cos (\alpha + \beta)$
(c) $2 \sin (\alpha + \beta)$ (d) $2 \cos (\alpha + \beta)$

27. If $x_i > 0$ for $1 \le i \le n$ and $x_1 + x_2 + \ldots + x_n = \pi$ then the greatest value of the sum $\sin x_1 + \sin x_2 + \sin x_3 + \ldots + \sin x_n =$
(a) n (b) π
(c) $n \sin \left(\frac{\pi}{n} \right)$ (d) 0

28. If $A = \sin^8 \theta + \cos^{14} \theta$, then for all values of θ,
(a) $A \ge 1$ (b) $0 < A < 1$
(c) $1 < 2a < 3$ (d) None of these

29. If $\sin\alpha = -3/5$ and lies in the third quadrant, then the value of $\cos\alpha/2$ is
(a) $1/5$ (b) $-1/\sqrt{10}$
(c) $-1/5$ (d) $1/\sqrt{10}$

30. The values of θ $(0 < \theta < 360^\circ)$ satisfying $\operatorname{cosec}\theta + 2 = 0$ are
(a) $210^\circ, 300^\circ$ (b) $240^\circ, 300^\circ$
(c) $210^\circ, 240^\circ$ (d) $210^\circ, 330^\circ$

31. If $\sin^3 x \sin 3x = \sum_{m=0}^{6} c_m \cos^m x$
where $c_0, c_1, c_2, \ldots, c_6$ are constants, then
(a) $c_0 + c_2 + c_4 + c_5 = 0$
(b) $c_1 + c_3 + c_5 = 6$
(c) $2c_2 + 3c_6 = 0$
(d) $c_4 + 2c_6 = 0$

32. If P is a point on the altitude AD of the triangle ABC such that $\angle CBP = B/3$, then AP is equal to
(a) $2a \sin(C/3)$ (b) $2b \sin(A/3)$
(c) $2c \sin(B/3)$ (d) $2c \sin(C/3)$

33. For what and only what values of α lying between 0 and π is the inequality
$\sin\alpha\cos^3\alpha > \sin^3\alpha\cos\alpha$ valid ?
(a) $\alpha \in (0, \pi/4)$ (b) $\alpha \in (0, \pi/2)$
(c) $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) None of these

34. Which of the following is correct
(a) $\sin 1^\circ > \sin 1$ (b) $\sin 1^\circ < \sin 1$
(c) $\sin 1^\circ = \sin 1$ (d) $\sin 1^\circ = \frac{\pi}{180} \sin 1$

35. If α, β, γ do not differ by a multiple of π and if $\frac{\cos(\alpha+\theta)}{\sin(\beta+\gamma)} = \frac{\cos(\beta+\theta)}{\sin(\gamma+\alpha)}$
$= \frac{\cos(\gamma+\theta)}{\sin(\alpha+\beta)} = k$ Then k equals
(a) ± 2 (b) $\pm 1/2$
(c) 0 (d) ± 1

36. If the expression $\frac{A\cos(\theta+\alpha) + B\sin(\theta+\beta)}{A'\sin(\theta+\alpha) + B'\cos(\theta+\beta)}$ retain the same value for all 'θ' then
(a) $(AA' - BB')\sin(\alpha-\beta) = (A'B - AB')$
(b) $AA' + BB' = (A'B + AB')\sin(\alpha-\beta)$
(c) $AA' - BB' = (A'B - AB')\sin(\alpha-\beta)$
(d) None of these

37. If $0 < x < \pi/2$, then
(a) $\cos x > 1 - \frac{2}{\pi}x$
(b) $\cos x < 1 - \frac{2}{\pi}x$
(c) $\cos x > \frac{2}{\pi}x$
(d) $\cos x < \frac{2}{\pi}x$

38. The minimum and maximum value of $ab\sin x + b\sqrt{(1-a^2)}\cos x + c$ $(|a| < 1,\ b > 0)$ respectively are
(a) $\{b-c, b+c\}$ (b) $\{b+c, b-c\}$
(c) $\{c-b, b+c\}$ (d) None of these

39. If $\cos x = \tan y$, $\cos y = \tan z$ and $\cos z = \tan x$ then $\sin x$ equals
(a) $\sin y$ (b) $\sin z$
(c) $2\sin 18^\circ$ (d) $\sin(y+z)$

40. The value of the expression
$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{10\pi}{7} - \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is
(a) 0 (b) $-\frac{1}{4}$
(c) $\frac{1}{4}$ (d) $-\frac{1}{8}$

41. The value of $\sum_{r=1}^{18} \cos^2(5r)^\circ$ is, where x° denotes the degrees
(a) 0 (b) 7/2
(c) 17/2 (d) 25/2

42. If $4n\alpha = \pi$ then the numerical value of $\tan\alpha\tan 2\alpha\tan 3\alpha \ldots \tan(2n-1)\alpha =$
(a) -1 (b) 0
(c) 1 (d) 2

43. If $\tan\alpha$ is an integral solution of the equation $4x^2 - 16x + 15 < 0$ and $\cos\beta$ is the slope of the bisector of the angle in the first quadrant between the x and y axes then the value of $\sin(\alpha+\beta) : \sin(\alpha-\beta) =$
(a) -1 (b) 0
(c) 1 (d) 2

44. If $2\cos\theta+\sin\theta = 1$ then $7\cos\theta+6\sin\theta$ equals

(a) 1 or 2 (b) 2 or 3
(c) 2 or 4 (d) 2 or 6

45. The value of $2\text{ ver}\sin A - \text{ver}\sin^2 =$

(a) $\cos^2 A$ (b) $\sin^2 A$
(c) $\cos 2A$ (d) $\sin 2A$

46. The ratio of the greatest value of $2-\cos x+\sin^2 x$ to its least value is

(a) 1/4 (b) 9/4
(c) 13/4 (d) None of these

47. If $\cos x+\sin x = a\left(-\frac{\pi}{2}<x<-\frac{\pi}{4}\right)$ then $\cos 2x =$

(a) a^2 (b) $a\sqrt{2-a}$
(c) $a\sqrt{2+a}$ (d) $a\sqrt{2-a^2}$

48. Expression $2^{\sin\theta}+2^{-\cos\theta}$ is minimum when $\theta =$ ..... and its minimum value is

(a) $2n\pi+\frac{\pi}{4}, n\in I,$
(b) $2n\pi+\frac{7\pi}{4}, n\in I, 2^{1-1/\sqrt{2}}$
(c) $n\pi\pm\pi/4, n\in I, 2^{1-1/\sqrt{2}}$
(d) None of these

49. If in a triangle ABC, $\cos 3A+\cos 3B+\cos 3C = 1$, then one angle must be exactly equal to

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$
(c) π (d) $\frac{4\pi}{3}$

50. In a triangle ABC, angle A is greater than B. If the measures of angle A and B satisfy the equation $3\sin x-4\sin 3x-k=0$, $0<k<1$, then the measure of angle C is

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{3}$
(c) $\frac{2\pi}{3}$ (d) $\frac{4\pi}{3}$

51. If $\tan x\tan y = a$ and $x+y=\frac{\pi}{6}$, then $\tan x$ and $\tan y$ satisfy the equation

(a) $x^2-\sqrt{3}(1-a)x+a=0$
(b) $\sqrt{3}x^2-(1-a)x+a\sqrt{3}=0$
(c) $x^2+\sqrt{3}(1+a)x-a=0$
(d) $\sqrt{3}x^2+(1+a)x-a\sqrt{3}=0$

52. If $\frac{\sin^3\theta-\cos^3\theta}{\sin\theta-\cos\theta}-\frac{\cos\theta}{\sqrt{(1+\cot^2\theta)}}-2\tan\theta\cot\theta=-1$, $\theta\in[0, 2\pi]$, then

(a) $\theta\in(0,\pi/2)-\{\pi/4\}$
(b) $\theta\in\left(\frac{\pi}{2},\pi\right)-\{3\pi/4\}$
(c) $\theta\in\left(\pi,\frac{3\pi}{2}\right)-\{5\pi/4\}$
(d) $\theta\in(0,\pi)-\{\pi/4,\pi/2\}$

53. If $\sin\alpha=\frac{336}{625}$ and $450°<\alpha<540°$, then $\sin\alpha/4$ is equal to

(a) $\frac{1}{5\sqrt{2}}$ (b) $\frac{7}{25}$
(c) $\frac{4}{5}$ (d) $\frac{3}{5}$

54. If $\sin(\theta+\alpha)=a$ and $\sin(\theta+\beta)=b$, then $\cos 2(\alpha-\beta)-4ab\cos(\alpha-\beta)$ is equal to

(a) $1-a^2-b^2$ (b) $1-2a^2-2b^2$
(c) $2+a^2-b^2$ (d) $2-a^2-b^2$

55. The expression $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ is equal to

(a) 0 (b) -1
(c) 1 (d) 3

MULTIPLE CHOICE –II

Each question in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

56. If $0<x<\frac{\pi}{2}$ and $\sin^n x+\cos^n x\geq 1$ then

(a) $n\in[2,\infty)$
(b) $n\in(-\infty, 2]$
(c) $n\in[-1, 1]$
(d) none of these

57. If $\frac{\tan x}{1} = \frac{\tan y}{2} = \frac{\tan z}{3}$ $(\neq 0)$ and $x + y + z = \pi$ then

(a) maximum value of $\tan x + \tan y + \tan z$ is 6

(b) minimum value of $\tan x + \tan y + \tan z$ is -6

(c) $\tan x = \pm 1$, $\tan y = \pm 2$, $\tan z = \pm 3$

(d) $\tan x + \tan y + \tan z = 0 \ \forall \ x, y, z \in R$

58. If $3 \sin \beta = \sin (2\alpha + \beta)$ then

(a) $[\cot \alpha + \cot (\alpha + \beta)] [\cot \beta - 3 \cot (2\alpha + \beta)] = 6$

(b) $\sin \beta = \cos (\alpha + \beta) \sin \alpha$

(c) $2 \sin \beta = \sin (\alpha + \beta) \cos \alpha$

(d) $\tan (\alpha + \beta) = 2 \tan \alpha$

59. Let $P_n(u)$ be a polynomial in u of degree n. Then, for every positive integer n, $\sin 2nx$ is expressible is

(a) $P_{2n} (\sin x)$

(b) $P_{2n} (\cos x)$

(c) $\cos x \, P_{2n-1} (\sin x)$

(d) $\sin x \, P_{2n-1} (\cos x)$

60. If $\sin^4 \alpha \cos^2 \alpha = \sum_{k=0}^{3} C_k \cos 2k\alpha$. then

$C_1 + C_3 =$ ...., $C_0 + C_2 =$ .... and $C_1 + C_2 + C_3 + C_0 =$ .....

(a) 0, 1, 2 (b) 0, 0, 0,

(c) 0, 2, 3 (d) 0, –1, 2

61. If $\cos \alpha = \frac{3}{5}$ and $\cos \beta = \frac{5}{13}$, then

(a) $\cos (\alpha + \beta) = \frac{33}{65}$

(b) $\sin (\alpha + \beta) = \frac{56}{65}$

(c) $\sin^2 \left(\frac{\alpha - \beta}{2} \right) = \frac{1}{65}$

(d) $\cos (\alpha - \beta) = \frac{63}{65}$

62. The equation $\sin^6 x + \cos^6 x = a^2$ has real solutions if

(a) $a \in (-1, 1)$ (b) $a \in \left(-1, -\frac{1}{2} \right)$

(c) $a \in \left(-\frac{1}{2}, \frac{1}{2} \right)$ (d) $a \in \left(\frac{1}{2}, 1 \right)$

63. If $\tan \alpha$ and $\tan \beta$ are the roots of the equation $x^2 + px + q = 0$ $(p \neq 0)$, then

(a) $\sin^2 (\alpha + \beta) + p \sin (\alpha + \beta) \cos (\alpha + \beta) + q \cos^2 (\alpha + \beta) = q$

(b) $\tan (\alpha + \beta) = p/(q - 1)$

(c) $\cos (\alpha + \beta) = 1 - q$

(d) $\sin (\alpha + \beta) = -p$

64. If $\sin \theta + \sin \phi = a$ and $\cos \theta + \cos \phi = b$, then

(a) $\cos \left(\frac{\theta - \phi}{2} \right) = \pm \frac{1}{2} \sqrt{(a^2 + b^2)}$

(b) $\cos \left(\frac{\theta - \phi}{2} \right) = \pm \frac{1}{2} \sqrt{(a^2 - b^2)}$

(c) $\tan \left(\frac{\theta - \phi}{2} \right) = \pm \sqrt{\left(\frac{4 - a^2 - b^2}{a^2 + b^2} \right)}$

(d) $\cos (\theta - \phi) = \frac{a^2 + b^2 - 2}{2}$

65. Which of the following statements are possible, a, b, m and n being non-zero real numbers ?

(a) $4 \sin^2 \theta = 5$

(b) $(a^2 + b^2) \cos \theta = 2ab$

(c) $(m^2 + n^2) \operatorname{cosec} \theta = m^2 - n^2$

(d) $\sin \theta = 2{\cdot}375$

66. Let $f_n (\theta) = \tan \frac{\theta}{2} (1 + \sec \theta) (1 + \sec 2\theta) (1 + \sec 4\theta) \ldots (1 + \sec 2^n \theta)$. Then

(a) $f_2 \left(\frac{\pi}{16} \right) = 1$ (b) $f_3 \left(\frac{\pi}{32} \right) = 1$

(c) $f_4 \left(\frac{\pi}{64} \right) = 1$ (d) $f_5 \left(\frac{\pi}{128} \right) = 1$

67. The set of values of $\lambda \in R$ such that $\tan^2 \theta + \sec \theta = \lambda$ holds for some θ is

(a) $(-\infty, 1]$ (b) $(-\infty, -1]$

(c) ϕ (d) $[-1, \infty)$

68. If the mapping $f(x) = ax + b$, $a < 0$ maps $[-1, 1]$ onto $[0, 2]$ then for all values of θ, $A = \cos^2 \theta + \sin^4 \theta$ is

(a) $f\left(\frac{1}{4} \right) < A < f(0)$

(b) $f(0) < A < f(-2)$

(c) $f\left(\frac{1}{3} \right) < A < f(0)$

(d) $f(-1) < A < f(-2)$

69. For $0 < \phi < \pi/2$ if

$$x = \sum_{n=0}^{\infty} \cos^{2n}\phi,\ y = \sum_{n=0}^{\infty} \sin^{2n}\phi,$$

$$z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi, \text{ then}$$

(a) $xyz = xz + y$
(b) $xyz = xy + z$
(c) $xyz = x + y + z$
(d) $xyz = yz + x$

70. If $\tan x = \dfrac{2b}{a-c}$, $(a \neq c)$

$$y = a\cos^2 x + 2b \sin x \cos x + c \sin^2 x$$
$$z = a\sin^2 x - 2b \sin x \cos x + c \cos^2 x, \text{ then}$$

(a) $y = z$
(b) $y + z = a + c$
(c) $y - z = a - c$
(d) $y - z = (a-c)^2 + 4b^2$

71. $\left(\dfrac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\dfrac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ (n, even or odd) $=$

(a) $2\tan^n\left(\dfrac{A-B}{2}\right)$ (b) $2\cot^n\left(\dfrac{A-B}{2}\right)$
(c) 0 (d) None of these

72. $\dfrac{3 + \cot 76^\circ \cot 16^\circ}{\cot 76^\circ + \cot 16^\circ} =$

(a) $\tan 16^\circ$ (b) $\cot 76^\circ$
(c) $\tan 46^\circ$ (d) $\cot 44^\circ$

73. In a triangle $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$, then the values of $\tan A$, $\tan B$ and $\tan C$ are

(a) 1, 2, 3 (b) 2, 1, 3
(c) 1, 2, 0 (d) None of these

74. If $\cos\theta = \dfrac{a\cos\phi + b}{a + b\cos\phi}$, then $\tan\theta/2 =$

(a) $\sqrt{\left(\dfrac{a-b}{a+b}\right)} \tan\phi/2$
(b) $\sqrt{\left(\dfrac{a+b}{a-b}\right)} \cos\phi/2$
(c) $\sqrt{\left(\dfrac{a-b}{a+b}\right)} \sin\phi/2$
(d) None of these

75. The value of $e^{\log_{10}\tan 1^\circ + \log_{10}\tan 2^\circ + \log_{10}\tan 3^\circ + \dots + \log_{10}\tan 89^\circ}$ is

(a) 0 (b) e
(c) $1/e$ (d) None of these

76. If in ΔABC, $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$ then $\sin^2 A : \sin^2 B : \sin^2 C$ is

(a) 8 : 9 : 5 (b) 8 : 5 : 9
(c) 5 : 9 : 8 (d) 5 : 8 : 5

77. If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$, then

(a) $\sin\theta\cos\theta = \dfrac{1}{x}$
(b) $\sin\theta\tan\theta = y$
(c) $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$
(d) $(x^2y)^{1/3} + (xy^2)^{1/3} = 1$

78. If $\dfrac{x}{y} = \dfrac{\cos A}{\cos B}$ where $A \neq B$ then

(a) $\tan\left(\dfrac{A+B}{2}\right) = \dfrac{x\tan A + y\tan B}{x+y}$
(b) $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{x\tan A - y\tan B}{x+y}$
(c) $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{y\sin A + x\sin B}{y\sin A - x\sin B}$
(d) $x\cos A + y\cos B = 0$

79. If $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then

(a) $\sin\alpha - \cos\alpha = \pm\sqrt{2}\sin\theta$
(b) $\sin\alpha + \cos\alpha = \pm\sqrt{2}\cos\theta$
(c) $\cos 2\theta = \sin 2\alpha$
(d) $\sin 2\theta + \cos 2\alpha = 0$

80. Let $0 < \theta < \pi/2$ and $x = X\cos\theta + Y\sin\theta$, $y = X\sin\theta - Y\cos\theta$ such that $x^2 + 4xy + y^2 = aX^2 + bY^2$, where a, b are constants. Then

(a) $a = -1, b = 3$
(b) $\theta = \pi/4$
(c) $a = 3, b = -1$
(d) $\theta = \pi/3$

Practice Test

MM : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. Minimum value of $4x^2 - 4x \mid \sin\theta \mid - \cos^2\theta$ is
(a) –2 (b) –1
(c) –1/2 (d) 0

2. For any real θ, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is
(a) 1 (b) $1 + \sin^2 1$
(c) $1 + \cos^2 1$ (d) does not exists

3. If in a triangle ABC, CD is the angular bisector of the angle ACB then CD is equal to
(a) $\frac{a+b}{2ab}\cos(C/2)$
(b) $\frac{a+b}{ab}\cos(C/2)$
(c) $\frac{2ab}{a+b}\cos(C/2)$
(d) $\frac{b\sin A}{\sin(B + C/2)}$

4. If $\cos^4\theta\sec^2\alpha$, $\frac{1}{2}$ and $\sin^4\theta\,\text{cosec}^2\alpha$ are in A.P. then $\cos^8\theta\sec^6\alpha$, $\frac{1}{2}$ and $\sin^8\theta\,\text{cosec}^6\alpha$ are in
(a) A. P. (b) G. P
(c) H. P. (d) None of them

5. Given that
$$(1+\sqrt{(1+x)})\tan x = 1+\sqrt{(1-x)}$$
Then $\sin 4x$ is equal to
(a) $4x$ (b) $2x$
(c) x (d) None of these

6. If $\tan\theta = n\tan\phi$, then maximum value of $\tan^2(\theta - \phi)$ is
(a) $\frac{(n+1)^2}{4n}$ (b) $\frac{(n-1)^2}{4n}$
(c) $\frac{(2n+1)^2}{4n}$ (d) $\frac{(2n-1)^2}{4n}$

7. If $\mid \tan A \mid < 1$, and $\mid A \mid$ is acute then $\frac{\sqrt{1+\sin 2A}+\sqrt{1-\sin 2A}}{\sqrt{1+\sin 2A}-\sqrt{1-\sin 2A}}$ is equal to
(a) $\tan A$ (b) $-\tan A$
(c) $\cot A$ (d) $-\cot A$

8. The maximum value of the expression $\mid \sqrt{(\sin^2 x + 2a^2)} - \sqrt{(2a^2 - 1 - \cos^2 x)} \mid$, where a and x are real numbers is
(a) $\sqrt{3}$ (b) $\sqrt{2}$
(c) 1 (d) $\sqrt{5}$

9. If $a_{n+1} = \sqrt{\frac{1}{2}(1 + a_n)}$, then $\cos\left(\frac{\sqrt{1-a_0^2}}{a_1 a_2 a_3 \ldots \text{to } \infty}\right)$ is equal to
(a) 1 (b) – 1
(c) a_0 (d) $\frac{1}{a_0}$

10. If in ΔABC, $\angle A = 90°$ and c, $\sin B$, $\cos B$ are rational numbers then
(a) a is the rational
(b) a is irrational
(c) b is rational
(d) b is irrational

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answer

Multiple Choice– I

1. (a)	**2.** (a)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (a)
7. (a)	**8.** (a)	**9.** (a)	**10.** (c)	**11.** (c)	**12.** (b)
13. (a)	**14.** (d)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (a)
19. (d)	**20.** (c)	**21.** (b)	**22.** (a)	**23.** (b)	**24.** (d)
25. (b)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (b)	**30.** (d)
31. (a)	**32.** (c)	**33.** (a)	**34.** (b)	**35.** (d)	**36.** (c)
37. (a)	**38.** (d)	**39.** (c)	**40.** (c)	**41.** (c)	**42.** (c)
43. (c)	**44.** (d)	**45.** (b)	**46.** (c)	**47.** (d)	**48.** (b)
49. (b)	**50.** (c)	**51.** (b)	**52.** (d)	**53.** (c)	**54.** (b)
55. (c)					

Multiple Choice–II

56. (a),	**57.** (a), (b), (c)	**58.** (a), (b), (c)	**59.** (c), (d)	**60.** (b)	**61.** (b), (c), (d)
62. (b), (d)	**63.** (a), (b)	**64.** (a), (c), (d)	**65.** (b), (d)	**66.** (a), (b), (c), (d)	
67. (d)	**68.** (a)	**69.** (b), (c)	**70.** (b), (c)	**71.** (b), (c)	**72.** (c), (d)
73. (a), (b)	**74.** (a)	**75.** (d)	**76.** (b), (d)	**77.** (a), (b), (c)	**78.** (a), (b) (c)
79. (a), (b), (c), (d)		**80.** (b), (c).			

Practice Test

1. (b)	**2.** (b)	**3.** (c), (d)	**4.** (a)	**5.** (c)	**6.** (b)
7. (b)	**8.** (b)	**9.** (c)	**10.** (a), (c).		

28

TRIGONOMETRIC EQUATIONS

§ 28.1. Reduce any trigonometric equation to one of the following forms

(i) If $\sin\theta = \sin\alpha$ or $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ then

$$\theta = n\pi + (-1)^n \alpha, \quad n \in I$$

(ii) If $\cos\theta = \cos\alpha$ or $\sec\theta = \sec\alpha$ then

$$\theta = 2n\pi \pm \alpha, \quad n \in I$$

(iii) If $\tan\theta = \tan\alpha$ or $\cot\theta = \cot\alpha$, then

$$\theta = n\pi + \alpha, \quad n \in I$$

(iv) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$.

then $\theta = n\pi \pm \alpha, \quad n \in I$

(v) If $\cos\theta = 0$ then $\theta = n\pi + \frac{\pi}{2}, \quad n \in I$

(vi) If $\cos\theta = 1$ then $\theta = 2n\pi, \quad n \in I$

(vii) If $\cos\theta = -1$ then $\theta = 2n\pi + \pi, \quad n \in I$

(viii) If $\sin\theta = 0$, then $\theta = n\pi, \quad n \in I$

(ix) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2}. \quad n \in I$

(x) If $\sin\theta = -1$, then $\theta = 2n\pi - \frac{\pi}{2}, \quad n \in I$

(xi) Equation of the type of

$$a\cos\theta + b\sin\theta = c \quad \text{...(1)}$$

then put $a = r\cos\alpha,\ b = r\sin\alpha$

$\therefore \quad r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

then equation (1) reduces to

$$r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = c$$

$$\Rightarrow \quad \cos(\theta - \alpha) = \frac{c}{r}$$

$$\Rightarrow \quad (\theta - \alpha) = 2n\pi \pm \cos^{-1}\left(\frac{c}{r}\right)$$

$$\Rightarrow \quad \theta = \alpha + 2n\pi \pm \cos^{-1}\left(\frac{c}{r}\right),\ n \in I$$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. The number of values of x for which $\sin 2x + \cos 4x = 2$ is
(a) 0 (b) 1
(c) 2 (d) infinite

2. The number of solutions of the equation $x^3 + x^2 + 4x + 2 \sin x = 0$ in $0 \le x \le 2\pi$ is
(a) zero (b) one
(c) two (d) four

3. Let α, β be any two positive values of x for which $2 \cos x$, $|\cos x|$ and $1 - 3 \cos^2 x$ are in G.P. The minimum value of $|\alpha - \beta|$ is
(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/2$ (d) None of these

4. The number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is
(a) 0 (b) 1
(c) 2 (d) 3

5. If $2 \tan^2 x - 5 \sec x$ is equal to 1 for exactly 7 distinct values of $x \in \left[0, \frac{n\pi}{2}\right]$, $n \in N$, then the greatest value of n is
(a) 6 (b) 12
(c) 13 (d) 15

6. The general solution of the trigonometrical equation $\sin x + \cos x = 1$ for $n = 0, \pm 1, \pm 2, \ldots$ is given by
(a) $x = 2n\pi$
(b) $x = 2n\pi + \pi/2$
(c) $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$
(d) None of these

7. The solution set of $(2 \cos x - 1)(3 + 2 \cos x) = 0$ in the interval $0 \le x \le 2x$ is
(a) $\left\{\frac{\pi}{3}\right\}$
(b) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$
(c) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}, \cos^{-1}\left(-\frac{3}{2}\right)\right\}$
(d) None of these

8. The smallest positive root of the equation $\tan x - x = 0$, lies in
(a) $(0, \pi/2)$ (b) $\left(\frac{\pi}{2}, \pi\right)$
(c) $\left(\pi, \frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2}, 2\pi\right)$

9. The number of solutions of the equation $\sin^5 x - \cos^5 x = \frac{1}{\cos x} - \frac{1}{\sin x}$ $(\sin x \ne \cos x)$ is
(a) 0 (b) 1
(c) infinite (d) none of these

10. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$, where x is a variable, has real roots. Then the interval of p may be any one of the followings
(a) $(0, 2\pi)$ (b) $(-\pi, 0)$
(c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $(0, \pi)$

11. The number of solutions of the equation $2(\sin^4 2x + \cos^4 2x) + 3 \sin^2 x \cos^2 x = 0$ is
(a) 0 (b) 1
(c) 2 (d) 3

12. $\cos 2x + a \sin x = 2a - 7$ possesses a solution for
(a) all a (b) $a > 6$
(b) $a < 2$ (d) $a \in [2, 6]$

13. The complete solution of the equation $7 \cos^2 x + \sin x \cos x - 3 = 0$ is given by
(a) $n\pi + \pi/2$ $(n \in I)$
(b) $n\pi - \pi/4$ $(n \in I)$
(c) $n\pi + \tan^{-1}(4/3)$ $(n \in I)$
(d) $n\pi + \frac{3\pi}{4}$, $k\pi + \tan^{-1}(4/3)$ $(k, n \in I)$

14. If $0 \le x \le \pi$ and $81^{\sin^2 x} + 81^{\cos^2 x} = 30$ then x is equal to
(a) $\pi/6$ (b) $\pi/2$
(c) π (d) $\pi/4$

15. If $1 + \sin\theta + \sin^2\theta + \ldots \infty = 4 + 2\sqrt{3}$, $0 < \theta < \pi$, $\theta \ne \pi/2$ then

(a) $\theta = \pi/6$ (b) $\theta = \pi/3$
(c) $\theta = \pi/3$ or $\pi/6$ (d) $\theta = \pi/3$ or $2\pi/3$

16. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value(s) of $\cos(\theta - \pi/4)$ is (are)

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\frac{1}{2\sqrt{2}}$ (d) None of these

17. The equation $a\sin x + b\cos x = c$ where $|c| > \sqrt{a^2 + b^2}$ has

(a) One solution
(b) Two solutions
(c) No solution
(d) Infinite number of solutions

18. The most general values of x for which

$$\sin x + \cos x = \min_{a \in R} \{1, a^2 - 4a + 6\}$$

are given by

(a) $2n\pi$
(b) $2n\pi + \frac{\pi}{2}$
(c) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$
(d) None of these

19. If $f(x) = \sin x + \cos x$. Then the most general solutions of $f(x) = \left[f\left(\frac{\pi}{10}\right)\right]$ are (where $[x]$ is the greatest integer less than or equal to x.)

(a) $2n\pi + \frac{\pi}{2}$, $n \in I$ (b) $n\pi$, $n \in I$
(c) $2n\pi \pm \frac{2\pi}{3}$, $n \in I$ (d) None of these

20. If $x \in [0, 2\pi]$, $y \in [0, 2\pi]$ and $\sin x + \sin y = 2$ then the value of $x + y$ is

(a) π (b) $\frac{\pi}{2}$
(c) 3π (d) None of these

21. The number of roots of the equation $x + 2\tan x = \pi/2$ in the interval $[0, 2\pi]$ is

(a) 1 (b) 2
(c) 3 (d) infinite

22. If $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$ and $x^2 + 4xy + y^2 = AX^2 + BY^2$, $0 \le \theta \le \pi/2$ then

(a) $\theta = \pi/6$ (b) $\theta = \pi/4$
(c) $A = -3$ (d) $B = 1$

23. The number of solutions of the equation $\cos(\pi\sqrt{x-4})\cos(\pi\sqrt{x}) = 1$ is

(a) None (b) One
(c) Two (d) More than two

24. The number of solutions of the equation

$$\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$$

(a) Forms an empty set
(b) is only one
(c) is only two
(d) is greater than 2

25. The solution of the equation $\log_{\cos x}\sin x + \log_{\sin x}\cos x = 2$ is given by

(a) $x = 2n\pi + \pi/4$ (b) $x = n\pi + \pi/2$
(c) $x = n\pi + \pi/8$ (d) None of these

26. The general value of θ such that $\sin 2\theta = \sqrt{3}/2$ and $\tan\theta = \frac{1}{\sqrt{3}}$ is given by

(a) $n\pi + \frac{7\pi}{6}$ (b) $n\pi \pm \frac{7\pi}{6}$
(c) $2n\pi + \frac{7\pi}{6}$ (d) None of these

27. Values of x and y satisfying the equation

$$\sin^7 y = |x^3 - x^2 - 9x + 9| + |x^3 - 4x - x^2 + 4| + \sec^2 2y + \cos^4 y$$ are

(a) $x = 1$, $y = n\pi$
(b) $x = 1$, $y = 2n\pi + \pi/2$
(c) $x = 1$, $y = 2n\pi$
(d) None of these

28. Number of real roots of the equation $\sec\theta + \operatorname{cosec}\theta = \sqrt{15}$ lying between 0 and 2π is

(a) 8 (b) 4
(c) 2 (d) 0

29. The solution of the equation

$$\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$$ is

(a) $x = \frac{n\pi}{4} + \frac{\pi}{8}$ (b) $x = n\pi + \frac{\pi}{4}$
(c) $x = 2n\pi + \frac{\pi}{2}$ (d) None of these

30. Solutions of the equation $|\cos x| = 2[x]$ are (where [.] denotes the greatest integer function)

(a) Nill (b) $x = \pm 1$

(c) $x = \pi/3$ (d) None of these

31. The general solution of the equation $\sin^{100} x - \cos^{100} x = 1$ is

(a) $2n\pi + \frac{\pi}{3}, n \in I$ (b) $n\pi + \frac{\pi}{2}, n \in I$

(c) $n\pi + \frac{\pi}{4}, n \in I$ (d) $2n\pi - \frac{\pi}{3}, n \in I$

32. The number of solutions of the equation $2^{\cos x} = |\sin x|$ in $[-2\pi, 2\pi]$ is

(a) 1 (b) 2

(c) 3 (d) 4

33. The general solution of the equation $2^{\cos 2x} + 1 = 3.2^{-\sin^2 x}$ is

(a) $n\pi$ (b) $n\pi + \pi$

(c) $n\pi - \pi$ (d) None of these

34. If $x \in (0, 1)$ the greatest root of the equation $\sin 2\pi x = \sqrt{2} \cos \pi x$ is

(a) 1/4 (b) 1/2

(c) 3/4 (d) None of these

35. If max $\{5 \sin \theta + 3 \sin (\theta - \alpha)\} = 7$, then the set of possible values of α is $\theta \in R$

(a) $\left\{x : x = 2n\pi \pm \frac{\pi}{3}, n \in I\right\}$

(b) $\left\{x : x = 2n\pi \pm \frac{2\pi}{3}, n \in I\right\}$

(c) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$

(d) None of these

MULTIPLE CHOICE –II

Each question in this part has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer (s).

36. $2 \sin x \cos 2x = \sin x$ if

(a) $x = n\pi + \pi/6 \ (n \in I)$

(b) $x = n\pi - \pi/6 \ (n \in I)$

(c) $x = n\pi \ (n \in I)$

(d) $x = n\pi + \pi/2 \ (n \in I)$

37. The equation

$$2 \sin \frac{x}{2} \cos^2 x - 2 \sin \frac{x}{2} \sin^2 x = \cos^2 x - \sin^2 x$$

has a root for which

(a) $\sin 2x = 1$ (b) $\sin 2x = -1$

(c) $\cos x = 1/2$ (d) $\cos 2x = -1/2$

38. $\sin x + \cos x = 1 + \sin x \cos x$, if

(a) $\sin (x + \pi/4) = \frac{1}{\sqrt{2}}$

(b) $\sin (x - \pi/4) = \frac{1}{\sqrt{2}}$

(c) $\cos (x + \pi/4) = \frac{1}{\sqrt{2}}$

(d) $\cos (x - \pi/4) = \frac{1}{\sqrt{2}}$

39. $\sin \theta + \sqrt{3} \cos \theta = 6x - x^2 - 11, 0 \le \theta \le 4\pi, x \in R$, holds for

(a) no value of x and θ

(b) one value of x and two values of θ

(c) two values of x and two values of θ

(d) two pairs of values of (x, θ)

40. The equation $\sin x = [1 + \sin x] + [1 - \cos x]$ has

(where $[x]$ is the greatest integer less than or equal to x)

(a) no solution in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) no solution in $\left[\frac{\pi}{2}, \pi\right]$

(c) no solution in $\left[\pi, \frac{3\pi}{2}\right]$

(d) no solution for $x \in R$

41. The set of all x in $(-\pi, \pi)$ satisfying $|4 \sin x - 1| < \sqrt{5}$ is given by

(a) $x \in \left(-\frac{\pi}{10}, \pi\right)$ (b) $x \in \left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$

(c) $x \in \left(-\pi, \frac{3\pi}{10}\right)$ (d) $x \in (-\pi, \pi)$

42. The solution of the inequality $\log_{1/2} \sin x > \log_{1/2} \cos x$ in $[0, 2\pi]$ is

(a) $x \in (0, \pi/2)$ (b) $x \in (0, \pi/8)$

(c) $x \in (0, \pi/4)$ (d) None of these

43. Solutions of the equation $\sin 7x + \cos 2x = -2$ are

(a) $x = \frac{2k\pi}{7} + \frac{3\pi}{14}, n, k \in I$

(b) $x = n\pi + \frac{\pi}{4}, n \in I$

(c) $x = n\pi + \pi/2, n \in I$
(d) None of these

44. The solutions of the system of equations $\sin x \sin y = \sqrt{3}/4$, $\cos x \cos y = \sqrt{3}/4$ are ....

(a) $x_1 = \frac{\pi}{3} + \frac{\pi}{2}(2n + k)$

(b) $y_1 = \frac{\pi}{6} + \frac{\pi}{2}(k - 2n)$

(c) $x_2 = \frac{\pi}{6} + \frac{\pi}{2}(2n + k)$

(d) $y_2 = \frac{\pi}{3} + \frac{\pi}{2}(k - 2n)$

45. $2\sin^2 x + \sin^2 2x = 2, -\pi < x < \pi$, then $x =$

(a) $\pm\pi/2$ (b) $\pm\pi/4$
(c) $+3\pi/4$ (d) None of these

46. The number of all possible triplets (x, y, z) such that
$(x + y) + (y + 2z)\cos 2\theta + (z - x)\sin^2\theta = 0$
for all θ is

(a) 0 (b) 1
(c) 3 (d) infinite

47. The number of solutions of
$\tan(5\pi\cos\alpha) = \cot(5\pi\sin\alpha)$ for α in $(0, 2\pi)$ is

(a) 7 (b) 14
(c) 21 (d) 28

48. The number of solution(s) of the equation
$\sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$
in the interval $[0, 2\pi]$ is/are

(a) No (b) One
(c) Two (d) Three

49. The most general values of x for which
$\sqrt{3}\sin x - \cos x = \min_{\lambda \in R}\{2, e^2, \pi, \lambda^2 - 4\lambda + 7\}$
are given by

(a) $2n\pi$

(b) $2n\pi + \frac{2\pi}{3}$

(c) $n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{6}$

(d) $n\pi + (-1)^{n+1}\frac{\pi}{4} - \frac{\pi}{3}$

50. The solution of the equation
$\cos^{103} x - \sin^{103} x = 1$ are

(a) $-\frac{\pi}{2}$ (b) 0

(c) $\frac{\pi}{2}$ (d) π

Practice Test

M.M. : 20 Time : 30 Min

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[$10 \times 2 = 20$]

1. If $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$, then possible values of x

(a) $n\pi$ or $n\pi + (-1)^n\pi/6$, $n \in I$

(b) $n\pi$ or $2n\pi + \frac{\pi}{2}$ or $n\pi + (-1)^n\frac{\pi}{6}$, $n \in I$

(c) $n\pi + (-1)^n\frac{\pi}{6}$, $n \in I$

(d) $n\pi$, $n \in I$

2. $\tan|x| = |\tan x|$ if

(a) $x \in (-\pi(2k + 1)/2, -\pi k]$
(b) $x \in [\pi k, \pi(2k + 1)/2)$
(c) $x \in (-\pi k, -\pi(2k - 1)/2)$
(d) $x \in (\pi(2k - 1)/2, \pi k)$, $k \in N$

3. The solution set of the inequality $\cos^2\theta < \frac{1}{2}$ is

(a) $\left\{\theta/(8n + 1)\frac{\pi}{4} < \theta < (8n + 3)\frac{\pi}{4}, n \in I\right\}$

(b) $\left\{\theta/(8n - 3)\frac{\pi}{4} < \theta < (8n - 1)\frac{\pi}{4}, n \in I\right\}$

(c) $\left\{\theta/(4n + 1)\frac{\pi}{4} < \theta < (4n + 3)\frac{\pi}{4}, n \in I\right\}$

(d) None of these

4. If $[y] = [\sin x]$ and $y = \cos x$ are two given equations, then the number of solutions, is :
([·] denotes the greatest integer function)

(a) 2 (b) 3
(c) 4

(d) Infinitely many solutions

(e) None of these

5. $\cos(\sin x) = \frac{1}{\sqrt{2}}$ then x must lie in the interval

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left[-\frac{\pi}{4}, 0\right]$

(c) $\left(\pi, \frac{3\pi}{2}\right)$ (d) $\left[\frac{\pi}{2}, \pi\right)$

6. A solution of the equation

$$(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$$

where θ lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is given by

(a) $\theta = 0$ (b) $\theta = \pi/3$

(c) $\theta = -\pi/3$ (d) $\theta = \pi/6$

7. The number of solutions of the equation $1 + \sin x \sin^2 x/2 = 0$ in $[-\pi, \pi]$ is

(a) zero (b) 1

(c) 2 (d) 3

8. The number of solutions of the equation

$$|\cot x| = \cot x + \frac{1}{\sin x} \quad (0 < x \le 2\pi) \text{ is}$$

(a) 0 (b) 1

(c) 2 (d) 3

9. The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$ are

(a) $-\pi/2, 0$ (b) $-\pi/2, 0, \pi/2$

(c) $\pi/2, 0$ (d) $0, \pi/4, \pi/2$

10. Number of solutions of the equations $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and $[y + [y]] = 2\cos x$, where [.] denotes the greatest integer function is

(a) 0 (b) 1

(c) 2 (d) infinite

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (a) **2.** (b) **3.** (d) **4.** (c) **5.** (d) **6.** (c)
7. (b) **8.** (c) **9.** (a) **10.** (d) **11.** (a) **12.** (d)
13. (d) **14.** (a) **15.** (d) **16.** (c) **17.** (c) **18.** (c)
19. (d) **20.** (a) **21.** (c) **22.** (b) **23.** (b) **24.** (b)
25. (a) **26.** (d) **27.** (b) **28.** (b) **29.** (a) **30.** (a)
31. (b) **32.** (d) **33.** (a) **34.** (c) **35.** (a)

Multiple Choice –II

36. (a), (b), (c) **37.** (a), (b), (c), (d) **38.** (a), (c), (d) **39.** (b), (d)
40. (a), (b), (c), (d) **41.** (b) **42.** (c) **43.** (a), (c)
44. (a), (b), (c), (d) **45.** (b), (c) **46.** (d) **47.** (b)
48. (a) **49.** (b) **50.** (a), (b)

Practice Test

1. (c), (d) **2.** (a), (b) **3.** (c) **4.** (d) **5.** (a), (d) **6.** (b), (c)
7. (a) **8.** (c) **9.** (b) **10.** (a)

29

INVERSE CIRCULAR FUNCTIONS

§ 29.1. Principal values for Inverse Circular Functions

$x < 0$	$x \geq 0$
$-\frac{\pi}{2} \leq \sin^{-1} x < 0$	$0 \leq \sin^{-1} x \leq \pi/2$
$\frac{\pi}{2} < \cos^{-1} x \leq \pi$	$0 \leq \cos^{-1} x \leq \pi/2$
$-\frac{\pi}{2} < \tan^{-1} x < 0$	$0 \leq \tan^{-1} x < \pi/2$
$\frac{\pi}{2} < \cot^{-1} x < \pi$	$0 < \cot^{-1} x \leq \pi/2$
$\frac{\pi}{2} < \sec^{-1} x \leq \pi$	$0 \leq \sec^{-1} x < \pi/2$
$-\frac{\pi}{2} \leq \operatorname{cosec}^{-1} x < 0$	$0 < \operatorname{cosec}^{-1} x \leq \pi/2$

Ex. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ not $\frac{2\pi}{3}$, $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ not $\frac{2\pi}{3}$.

§ 29.2. Some Results on Inverse Trigonometric Functions

(i) $\sin^{-1}(-x) = -\sin^{-1} x, \quad -1 \leq x \leq 1$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, \quad -1 \leq x \leq 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x, \quad x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, \quad x \in R$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x, \quad x \leq -1$ or $x \geq 1$

(vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x,$

(vii) $\sin^{-1} x + \cos^{-1} x = \pi/2, \quad -1 < x < 1$

(viii) $\tan^{-1} x + \cot^{-1} x = \pi/2, \quad x \in R$

(ix) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2, \quad x < -1$ or $x > 1$

(x) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right), \quad x \leq -1$ or $x \geq 1$

(xi) $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right), \quad x < -1$ or $x \geq 1$

(xii) $\cot^{-1} x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

(xiii) If $x > 0, \ y > 0, \ xy < 1$, then

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(xiv) If $x > 0,\ y > 0,\ xy > 1$, then

$$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(xv) If $x < 0,\ y < 0,\ xy > 1$, then

$$\tan^{-1} x + \tan^{-1} y = -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

(xvi) If $x > 0,\ y > 0,\ x^2+y^2 < 1$, then

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}]$$

(xvii) If $x > 0,\ y > 0,\ x^2+y^2 > 1$, then

$$\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}]$$

(xviii) If $0 < x,\ y < 1$ then

$$\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2}-y\sqrt{1-x^2}]$$

(xix) If $0 < x,\ y < 1$ then

$$\cos^{-1} x + \cos^{-1} y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$$

(xx) If $-1 < x,\ y < 0$ then

$$\cos^{-1} x + \cos^{-1} y = 2\pi - \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$$

(xxi) If $-1 \le x < y \le 1$ then

$$\cos^{-1} x - \cos^{-1} y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}].$$

(xxii) If $|x| < 1$ then

$$2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

(xxiii) If $|x| > 1$, then

$$\pi - 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. $\cos^{-1}\left[\cos\left(-\frac{17}{15}\pi\right)\right]$ is equal to
(a) $-\frac{17\pi}{15}$ (b) $\frac{17\pi}{15}$
(c) $\frac{2\pi}{15}$ (d) $\frac{13\pi}{15}$

2. $\tan\left[2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right]$
(a) $\frac{5}{4}$ (b) $\frac{5}{16}$
(c) $-\frac{7}{17}$ (d) $\frac{7}{17}$

3. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$,
then $\cos^{-1} x + \cos^{-1} y =$
(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{6}$ (d) π

4. If $x + 1/x = 2$, the principal value of $\sin^{-1} x$ is
(a) $\pi/4$ (b) $\pi/2$
(c) π (d) $3\pi/2$

5. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$ then $xy + yz + zx$ is equal to
(a) –3 (b) 0
(c) 3 (d) –1

6. The value of

$$\sin^{-1}\left[\cot\left(\sin^{-1}\sqrt{\left(\frac{2-\sqrt{3}}{4}\right)} + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}\sqrt{2}\right)\right]$$

(a) 0 (b) $\pi/4$
(c) $\pi/6$ (d) $\pi/2$

7. The number of real solutions of
$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{(x^2+x+1)} = \pi/2$ is
(a) zero (b) one

(c) two (d) infinite

8. A solution of the equation

$\tan^{-1}(1+x)+\tan^{-1}(1-x)=\pi/2$ is

(a) $x=1$ (b) $x=-1$
(c) $x=0$ (d) $x=\pi$

9. If x_1, x_2, x_3, x_4 are roots of the equation

$x^4-x^3\sin 2\beta+x^2\cos 2\beta-x\cos\beta-\sin\beta=0.$

then $\sum_{i=1}^{4}\tan^{-1}x_i=$

(a) β (b) $\pi/2-\beta$
(c) $\pi-\beta$ (d) $-\beta$

10. If $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}=\alpha$, then $x^2=$

(a) $\cos 2\alpha$ (b) $\sin 2\alpha$
(c) $\tan 2\alpha$ (d) $\cot 2\alpha$

11. If $x^2+y^2+z^2=r^2$, then

$\tan^{-1}\left(\dfrac{xy}{zr}\right)+\tan^{-1}\left(\dfrac{yz}{xr}\right)+\tan^{-1}\left(\dfrac{zx}{yr}\right)=$

(a) π (b) $\pi/2$
(c) 0 (d) None of these

12. $-\dfrac{2\pi}{5}$ is the principal value of

(a) $\cos^{-1}\left(\cos\dfrac{7\pi}{5}\right)$ (b) $\sin^{-1}\left(\sin\dfrac{7\pi}{5}\right)$
(c) $\sec^{-1}\left(\sec\dfrac{7\pi}{5}\right)$ (d) $\sin^{-1}\left(\sin\left(\dfrac{2\pi}{5}\right)\right)$

13. The value of

$\tan^{-1}(1)+\cos^{-1}(-1/2)+\sin^{-1}(-1/2)$

is equal to

(a) $\pi/4$ (b) $5\pi/12$
(c) $3\pi/4$ (d) $13\pi/12$

14. If $\sum_{i=1}^{2n}\sin^{-1}x_i=n\pi$ then $\sum_{i=1}^{2n}x_i$ is equal to

(a) n (b) $2n$
(c) $\dfrac{n(n+1)}{2}$ (d) None of these

15. The inequality $\sin^{-1}(\sin 5)>x^2-4x$ holds if

(a) $x=2-\sqrt{9-2\pi}$
(b) $x=2+\sqrt{9-2\pi}$
(c) $x\in(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi})$
(d) $x>2+\sqrt{9-2\pi}$

16. The sum of the infinite series

$\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\cot^{-1}32+\ldots$

is equal to

(a) π (b) $\pi/2$
(c) $\pi/4$ (d) None of these

17. $\sin\{\cot^{-1}(\tan\cos^{-1}x)\}$ is equal to

(a) x (b) $\sqrt{(1-x^2)}$
(c) $\dfrac{1}{x}$ (d) None of these

18. The value of $\tan^2(\sec^{-1}2)+\cot^2(\text{cosec}^{-1}3)$ is

(a) 13 (b) 15
(c) 11 (d) None of these

19. The equation $\sin^{-1}x=2\sin^{-1}a$ has a solution for

(a) all real values of a
(b) $a<1$
(c) $-1/\sqrt{2}<a<1/\sqrt{2}$
(d) $-1<a<1$

20. The number of real solutions of (x, y) where $|y|=\sin x$, $y=\cos^{-1}(\cos x)$, $-2\pi<x<2\pi$, is

(a) 2 (b) 1
(c) 3 (d) 4

21. The number of positive integral solutions of

$\tan^{-1}x+\cot^{-1}y=\tan^{-1}3$ is

(a) one
(b) two
(c) three
(d) four

22. The value of $\cos^{-1}(\cos 12)-\sin^{-1}(\sin 12)$ is

(a) 0 (b) π
(c) $8\pi-24$ (d) None of these

23. The smallest and the largest values of

$\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$, $0<x<1$ are

(a) $0, \pi$
(b) $0, \pi/4$
(c) $-\pi/4, \pi/4$
(d) $\pi/4, \pi/2$

24. If $-1<x<0$ then $\sin^{-1}x$ equals

(a) $\pi-\cos^{-1}(\sqrt{1-x^2})$

(b) $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

(c) $-\cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

(d) $\operatorname{cosec}^{-1} x$

25. The value of $\sin^{-1}(\sin 10)$ is

(a) 10 (b) $10-3\pi$

(c) $3\pi-10$ (d) None of these

26. If a, b are positive quantities and if

$$a_1 = \frac{a+b}{2},\ b_1 = \sqrt{a_1 b},$$

$$a_2 = \frac{a_1+b_1}{2},\ b_2 = \sqrt{a_2 b_1}$$ and so on then

(a) $a_\infty = \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}$

(b) $b_\infty = \dfrac{\sqrt{(b^2-a^2)}}{\cos^{-1}(a/b)}$

(c) $b_\infty = \dfrac{\sqrt{(a^2-b^2)}}{\cos^{-1}(b/a)}$

(d) None of these

27. $\tan^{-1}\left(\dfrac{c_1 x - y}{c_1 y + x}\right) + \tan^{-1}\left(\dfrac{c_2-c_1}{1+c_2c_1}\right) + \tan^{-1}\left(\dfrac{c_3-c_2}{1+c_3c_2}\right) + \ldots + \tan^{-1}\left(\dfrac{1}{c_n}\right) =$

(a) $\tan^{-1}(y/x)$ (b) $\tan^{-1}\dfrac{x}{y}$

(c) $-\tan^{-1}\left(\dfrac{x}{y}\right)$ (d) None of these

28. The value of

$$\sin^{-1}\left\{(\sin\pi/3)\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\} - \cos^{-1}\left\{\cos\pi/6\,\frac{x}{\sqrt{(x^2+k^2-kx)}}\right\}$$

$\left(\text{where } \dfrac{k}{2} < x < 2k,\ k > 0\right)$ is

(a) $\tan^{-1}\left(\dfrac{2x^2+xk-k^2}{x^2-2xk+k^2}\right)$

(b) $\tan^{-1}\left(\dfrac{x^2+2xk-k^2}{x^2-2xk+k^2}\right)$

(c) $\tan^{-1}\left(\dfrac{x^2+2xk-2k^2}{2x^2-2xk+2k^2}\right)$

(d) None of these

29. The value of $\tan\left\{\left(\cos^{-1}\left(-\dfrac{2}{7}\right)-\pi/2\right)\right\}$ is

(a) $\dfrac{2}{3\sqrt{5}}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{1}{\sqrt{5}}$ (d) $\dfrac{4}{\sqrt{5}}$

30. Sum infinite terms of the series

$$\cot^{-1}\left(1^2+\frac{3}{4}\right)+\cot^{-1}\left(2^2+\frac{3}{4}\right)+\cot^{-1}\left(3^2+\frac{3}{4}\right)+\ldots \text{ is}$$

(a) $\pi/4$ (b) $\tan^{-1}2$

(c) $\tan^{-1}3$ (d) None of these

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer(s). For each question, write the letters a, b, c, d corresponding to the correct answer(s).

31. The x satisfying $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$ are

(a) 0 (b) 1/2

(c) 1 (d) 2

32. If $2\tan^{-1}x+\sin^{-1}\dfrac{2x}{1+x^2}$ is independent of x then

(a) $x\in[1,+\infty)$ (b) $x\in[-1,1]$

(c) $x\in(-\infty,-1]$ (d) None of these

33. If $\dfrac{1}{2}<|x|<1$ then which of the following are real ?

(a) $\sin^{-1}x$ (b) $\tan^{-1}x$

(c) $\sec^{-1}x$ (d) $\cos^{-1}x$

34. $\sin^{-1}x>\cos^{-1}x$ holds for

(a) all values of x (b) $x\in(0, 1/\sqrt{2})$

(c) $x\in(1/\sqrt{2}, 1)$ (d) $x=0{\cdot}75$

35. $6\sin^{-1}(x^2-6x+8{\cdot}5)=\pi$, if

(a) $x=1$ (b) $x=2$
(c) $x=3$ (d) $x=4$

36. If $\cot^{-1}\left(\frac{n}{\pi}\right)>\left(\frac{\pi}{6}\right)$, $n\in N$, then the maximum value of n is
(a) 1 (b) 5
(c) 9 (d) None of these

37. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then x equals
(a) 0 (b) -1
(c) -2 (d) -3

38. The value of $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{1}{2r^2}\right)$ is
(a) $\pi/2$ (b) $\pi/4$
(c) π (d) 2π

39. The number of the positive integral solutions of
$$\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$$
(a) 1 (b) 2
(c) 3 (d) 4

40. Let $f(x)=e^{\cos^{-1}\sin(x+\pi/3)}$ then
(a) $f\left(\frac{8\pi}{9}\right)=e^{5\pi/18}$
(b) $f\left(\frac{8\pi}{9}\right)=e^{13\pi/18}$
(c) $f\left(-\frac{7\pi}{4}\right)=e^{\pi/12}$
(d) $f\left(-\frac{7\pi}{4}\right)=e^{11\pi/12}$

41. If $\alpha<\sin^{-1}x+\cos^{-1}x+\tan^{-1}x<\beta$, then
(a) $\alpha=0$ (b) $\beta=\pi/2$
(c) $\alpha=\pi/4$ (d) $\beta=\pi$

42. The greatest and least values of $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ are
(a) $\frac{\pi^3}{32}$ (b) $-\frac{\pi^3}{8}$
(c) $\frac{7\pi^3}{8}$ (d) $\frac{\pi}{2}$

43. $\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$ is
(a) $\pi/4$ (b) $\pi/2$
(c) π (d) 0

44. If $a_1, a_2, a_3, .., a_n$ is an A.P. with common difference d, then
$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$
is equal to
(a) $\frac{(n-1)d}{a_1+a_n}$ (b) $\frac{(n-1)d}{1+a_1a_n}$
(c) $\frac{nd}{1+a_1a_n}$ (d) $\frac{a_n-a_1}{a_n+a_1}$

45. If $\tan\theta+\tan\left(\frac{\pi}{3}+\theta\right)+\tan\left(-\frac{\pi}{3}+\theta\right)=k\tan 3\theta$, then the value of k is
(a) 1 (b) 1/3
(c) 3 (d) None of these

Practice Test

M.M : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).
[10 × 2 = 20]

1. The principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is
(a) π (b) $\pi/2$
(c) $\pi/3$ (d) $4\pi/3$

2. The sum of the infinite series
$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+\ldots\ldots+\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)+\ldots\ldots$$
(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) π

3. The solution of the equation $\sin[2\cos^{-1}[\cot(2\tan^{-1}x)]] = 0$ are

(a) ± 1

(b) $1 \pm \sqrt{2}$

(c) $-1 \pm \sqrt{2}$

(d) None of these

4. α, β and γ are the angles given by $\alpha = 2\tan^{-1}(\sqrt{2}-1)$, $\beta = 3\sin^{-1}(1/\sqrt{2}) + \sin^{-1}(-1/2)$ and $\gamma = \cos^{-1}(1/3)$ then

(a) $\alpha > \beta$ (b) $\beta > \gamma$

(c) $\gamma > \alpha$ (d) none of these

5. The number of distinct roots of the equation $A\sin^3 x + B\cos^3 x + C = 0$ no two of which differ by 2π is

(a) 3 (b) 4

(c) infinite (d) 6

6. If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$, where $[\cdot]$ denotes the greatest integer function, then x is given by the interval

(a) [tan sin cos 1, tan sin cos sin 1]

(b) (tan sin cos 1, tan sin cos sin 1)

(c) [– 1, 1]

(d) [sin cos tan 1, sin cos sin tan 1]

7. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = 3\pi/2$ then the value of $x^{100} + y^{100} + z^{100} - \frac{3}{x^{101} + y^{101} + z^{101}}$ is

(a) 0 (b) 1

(c) 2 (d) 3

8. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then the two curves $y = \cos x$ and $y = \sin 3x$ intersect at

(a) $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$

(b) $\left(-\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$

(c) $\left(\frac{\pi}{4}, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{\pi}{8}, -\cos\frac{\pi}{8}\right)$

(d) $\left(-\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

9. The solution of the inequality $(\cot^{-1}x)^2 - 5\cot^{-1}x + 6 > 0$ is

(a) (cot 3, cot 2)

(b) $(-\infty, \cot 3) \cup (\cot 2, \infty)$

(c) $(\cot 2, \infty)$

(d) None of these

10. Indicate the relation which is true

(a) $\tan|\tan^{-1}x| = |x|$

(b) $\cot|\cot^{-1}x| = x$

(c) $\tan^{-1}|\tan x| = |x|$

(d) $\sin|\sin^{-1}x| = |x|$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice –I

1. (d)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (a)
7. (c)	**8.** (c)	**9.** (b)	**10.** (b)	**11.** (b)	**12.** (b)
13. (c)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (a)	**18.** (c)
19. (c)	**20.** (c)	**21.** (b)	**22.** (c)	**23.** (b)	**24.** (b)
25. (c)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (a)	**30.** (b)

Multiple Choice –II

31. (a), (b)	**32.** (a)	**33.** (a), (b), (d)	**34.** (c), (d)	**35.** (b), (d)	**36.** (b)
37. (b)	**38.** (b)	**39.** (b)	**40.** (b), (c)	**41.** (a), (d)	**42.** (a), (c)
43. (c), (d)	**44.** (b)	**45.** (c)			

Practice Test

1. (a)	**2.** (c)	**3.** (a), (c)	**4.** (b), (c)	**5.** (d)	**6.** (a)
7. (c)	**8.** (a)	**9.** (b)	**10.** (a), (b), (d)		

30

PROPERTIES OF TRIANGLE

§ 30.1

Some important formulae relating the sides a, b, c and angles A, B, C of a triangle are :

1. Area of the ΔABC :

The area of $\Delta\, ABC$ (denoted by Δ or S) may be expressed in many ways as follows :

(i) $\Delta = \frac{1}{2}\, bc \sin A = \frac{1}{2}\, ca \sin B = \frac{1}{2}\, ab \sin C.$

(ii) $\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}\,,\; s = \dfrac{a+b+c}{2}$

(iii) $\Delta = \dfrac{a^2 \sin B \sin C}{2 \sin A} = \dfrac{b^2 \sin C \sin A}{2 \sin B} = \dfrac{c^2 \sin A \sin B}{2 \sin C}$

2. Sine Rule

In any $\Delta\, ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

3. Cosine Rule :

In any $\Delta\, ABC$,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}\,;\; \cos B = \frac{c^2 + a^2 - b^2}{2ca}\,;\; \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

4. Projection Rule

In any $\Delta\, ABC$,

$$a = b \cos C + c \cos B,\; b = c \cos A + a \cos C,\; c = a \cos B + b \cos A.$$

5. Tangent Rule (Nepier's Analogy) : In any $\Delta\, ABC$,

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot A/2,\quad \tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a} \cot B/2,$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot C/2$$

6. Trigonometrical Ratios of the Half-Angles of $\Delta\, ABC$

(i) $$\sin A/2 = \sqrt{\frac{(s-b)\,(s-c)}{bc}}\,,\; \sin B/2 = \sqrt{\frac{(s-c)\,(s-a)}{ca}}\,,$$

$$\sin C/2 = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$$

(ii) $$\cos A/2 = \sqrt{\frac{s\,(s-a)}{bc}}\,,\; \cos B/2 = \sqrt{\frac{s\,(s-b)}{ca}}\,,$$

$$\cos C/2 = \sqrt{\frac{s(s-c)}{ab}}$$

(iii) $\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan B/2 = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$$\tan C/2 = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

§ 30.2

The lengths of the radii of the circumcircle, the inscribed circle, and the escribed circles opposite to A, B, C will be denoted respectively by R, r, r_1, r_2, r_3.

1. Formulae for Circum-radius R

(i) $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

(ii) $R = \frac{abc}{4\Delta}$.

2. Formulae for In-radius r

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan A/2 = (s-b)\tan B/2 = (s-c)\tan C/2$

(iii) $r = \frac{a \sin B/2 \sin C/2}{\cos A/2} = \frac{b \sin C/2 \sin A/2}{\cos B/2} = \frac{c \sin A/2 \sin B/2}{\cos C/2}$

(iv) $r = 4R \sin A/2 \sin B/2 \sin C/2$

3 Formulae for Ex-radii r_1, r_2, r_3

(i) $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s \tan A/2$, $r_2 = s \tan B/2$, $r_3 = s \tan C/2$

(iii) $r_1 = \frac{a \cos B/2 \cos C/2}{\cos A/2}$, $r_2 = \frac{b \cos C/2 \cos A/2}{\cos B/2}$,

$r_3 = \frac{c \cos A/2 \cos B/2}{\cos C/2}$

(iv) $r_1 = 4R \sin A/2 \cos B/2 \cos C/2$, $r_2 = 4R \cos A/2 \sin B/2 \cos C/2$,
$r_3 = 4R \cos A/2 \cos B/2 \sin C/2$.

4. Orthocentre and Pedal triangle of any Triangle

Let ABC be any triangle and let the perpendiculars AD, BE and CF from vertices A, B and C on opposite BC, CA and AB respectively, meet at P. then P is the **orthocentre** of the ΔABC. (Fig. 30. 1) the triangle DEF, which is formed by joining the feet of these perpendiculars, is called the **pedal triangle** of ΔABC.

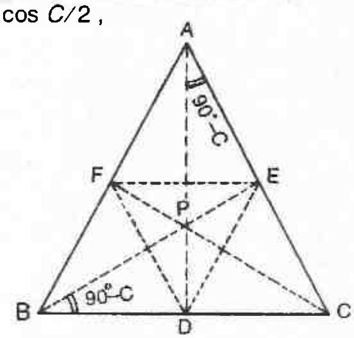


Fig. 30.1

5. The distances of the orthocentre from the Vertices and the Sides

(i) $PA = 2R \cos A$, $PB = 2R \cos B$, $PC = 2R \cos C$

(ii) $PD = 2R \cos B \cos C$, $PE = 2R \cos C \cos A$, $PF = 2R \cos A \cos B$.

6. Sides and Angles of the Pedal Triangle

(i) $EF = a\cos A,\quad DF = b\cos B,\quad DE = c\cos C$

(ii) $\angle EDF = 180^\circ - 2A,\quad \angle DEF = 180^\circ - 2B,\quad \angle EFD = 180^\circ - 2C$

7. Length of the Medians

If AD, BE and CF are the medians of the triangle ABC then

$$AD = \frac{1}{2}\sqrt{(2b^2 + 2c^2 - a^2)}$$

$$BE = \frac{1}{2}\sqrt{(2c^2 + 2a^2 - b^2)}$$

$$CF = \frac{1}{2}\sqrt{(2a^2 + 2b^2 - c^2)}$$

8. Distance between the Circumcentre and the Orthocentre

If O is the circumcentre and P is the orthocentre then

$$OP = R\sqrt{(1 - 8\cos A\cos B\cos C)}$$

9. Distance between the Circumcentre and the Incentre

If O is the circumcentre and I is the Incentre then

$$OI = R\sqrt{(1 - 8\sin A/2\sin B/2\sin C/2)} = \sqrt{(R^2 - 2Rr)}$$

10. Ptolemy's Theorem

In a cyclic quadrilateral $ABCD$.

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

11. Area of the Quadrilateral

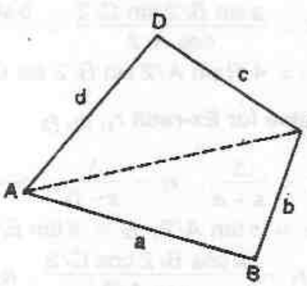


Fig. 30.2.

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos\alpha$$

Corollary I : If $d = 0$, then the quadrilateral becomes a triangle.

Corollary II : The quadrilateral , whose sides are given, has therefore the greatest area when it can be inscribed in a circle.

12. Regular Polygon

Let $A_1, A_2, \ldots A_n$ be a regular polygon of n sides each of length a.

(i) **Inscribed circle of a regular polygon of n sides :**

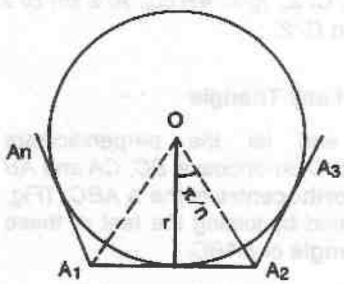


Fig. 30.3.

$$\text{Area} = nr^2\tan\pi/n$$

$$\text{Radius } r = \frac{a}{2}\cot\pi/n$$

(ii) Circumscribed circle of a regular polygon of n sides :

$$\text{Area} = \frac{nR^2}{2}\sin\frac{2\pi}{n}$$

$$\text{Radius} = R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

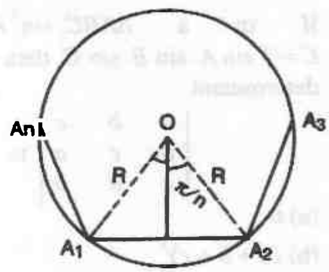


Fig. 30.4.

13. Some More Relations Regarding a Triangle :

(i) $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C.$

(ii) $a\cot A + b\cot B + c\cot C = 2(R + r)$

(iii) $r_1 + r_2 + r_3 = 4R + r$

(iv) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. In a trinagle $ABC, (a+b+c)(b+c-a) = k\,bc$ if

(a) $k < 0$ (b) $k > 6$
(c) $0 < k < 4$ (d) $k > 4$

2. If λ be the perimeter of the $\Delta\,ABC$ then $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$ is equal to

(a) λ (b) 2λ
(c) $\lambda/2$ (d) None of these

3. If the area of a triangle ABC is given by $\Delta = a^2 - (b-c)^2$ then $\tan A/2$ is equal to

(a) –1 (b) 0
(c) 1/4 (d) 1/2

4. The perimeter of a triangle ABC is 6 times the arithmetic mean of the sines of its angles. If the side a is 1 then $\angle A$ is

(a) 30° (b) 60°
(c) 90° (d) 120°

5. $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) =$

(a) $3abc$ (b) $3(a+b+c)$
(c) $abc(a+b+c)$ (d) 0

6. If $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and the side $a = 2$, then area of triangle is

(a) 1 (b) 2
(c) $\sqrt{3}/2$ (d) $\sqrt{3}$

7. If in a ΔABC, $\cos A + 2\cos B + \cos C = 2$, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

8. In a triangle $ABC, \angle B = \pi/3$ and $\angle c = \pi/4$ let D divide BC internally in the ratio 1 : 3. Then $\frac{\sin(\angle BAD)}{\sin(\angle CAD)}$ equals

(a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{3}$
(c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{\frac{2}{3}}$

9. If D is the mid point of side BC of a triangle ABC and AD is perpendicular to AC, then

(a) $3a^2 = b^2 - 3c^2$ (b) $3b^2 = a^2 - c^2$
(c) $b^2 = a^2 - c^2$ (d) $a^2 + b^2 = 5c^2$

10. If f, g, h are the internal bisectors of a ΔABC then $\frac{1}{f}\cos\frac{A}{2} + \frac{1}{g}\cos\frac{B}{2} + \frac{1}{h}\cos\frac{C}{2} =$

(a) $\frac{1}{a} + \frac{1}{b} - \frac{1}{c}$ (b) $\frac{1}{a} - \frac{1}{b} + \frac{1}{c}$
(c) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ (d) none of these

11. If a, b, c, d be the sides of a quadrilateral and $g(x) = f(f(f(x)))$ where $f(x) = \frac{1}{1-x}$. then $\frac{d^2}{a^2+b^2+c^2}$

(a) $> g(3)$ (b) $< g(3)$
(c) $> g(2)$ (d) $< g(4)$

12. If in a ΔABC, $\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A . \sin B . \sin C$, then the value of the determinant

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ is}$$

(a) 0
(b) $(a+b+c)^3$
(c) $(a+b+c)(ab+bc+ca)$
(d) None of these

13. In a ΔABC, if $r = r_2 + r_3 - r_1$, and $\angle A > \frac{\pi}{3}$ then the range of $\frac{s}{a}$ is equal to

(a) $\left(\frac{1}{2}, 2\right)$ (b) $\left(\frac{1}{2}, \infty\right)$
(c) $\left(\frac{1}{2}, 3\right)$ (d) $(3, \infty)$

14. If the bisector of angle A of triangle ABC makes an angle θ with BC. then $\sin\theta =$
(a) $\cos\left(\frac{B-C}{2}\right)$ (b) $\sin\left(\frac{B-C}{2}\right)$
(c) $\sin\left(B - \frac{A}{2}\right)$ (d) $\sin\left(C - \frac{A}{2}\right)$

15. With usual notations, if in a triangle ABC,

$$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$$

then $\cos A : \cos B : \cos C =$
(a) 7 : 19 : 25 (b) 19 : 7 : 25
(c) 12 : 14 : 20 (d) 19 : 25 : 20

16. If $b + c = 3a$, then the value of $\cot\frac{B}{2}\cot\frac{C}{2} =$
(a) 1 (b) 2
(c) $\sqrt{3}$ (d) $\sqrt{2}$

17. If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the sides are proportional to
(a) $1 : 1 : \sqrt{2}$ (b) $1 : \sqrt{2} : 1$
(c) $\sqrt{2} : 1 : 1$ (d) None of these

18. In an equilateral triangle, $R : r : r_2$ is equal to
(a) 1 : 1 : 1 (b) 1 : 2 : 3
(c) 2 : 1 : 3 (d) 3 : 2 : 4

19. If in a ΔABC, $a^2 + b^2 + c^2 = 8R^2$, where $R =$ circumradius, then the triangle is
(a) equilateral
(b) isosceles
(c) right angled
(d) None of these

20. Let $A_0 A_1 A_2 A_3 A_4 A_5$ be a regular hexagon inscribed in a circle of unit radius. The product of the length of the line segments A_0A_1, A_0A_2 and A_0A_4 is
(a) 3/4 (b) $3\sqrt{3}$
(c) 3 (d) $\frac{3\sqrt{3}}{2}$

21. If in a ΔABC, $r_1r_2 + r_2r_3 + r_3r_1 =$ (where r_1, r_2, r_3 are the ex-radii and $2s$ is the perimeter)
(a) s^2 (b) $2s^2$
(c) $3s^2$ (d) $4s^2$

22. In a ΔABC, the value of

$$\frac{a\cos A + b\cos B + c\cos C}{a+b+c} =$$

(a) $\frac{R}{r}$ (b) $\frac{R}{2r}$
(c) $\frac{r}{R}$ (d) $\frac{2r}{R}$

23. In a ΔABC, the sides a, b, c are the roots of the equation $x^3 - 11x^2 + 38x - 40 = 0$. Then $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ is equal to
(a) 1 (b) 3/4
(c) 9/16 (d) None of these

24. If the base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, then height of the triangle is equal to
(a) half the base (b) the base
(c) twice the base (d) four times the base

25. In a triangle ABC, $a = 4$, $b = 3$, $\angle A = 60°$. Then c is the root of the equation
(a) $c^2 - 3c - 7 = 0$ (b) $c^2 + 3c + 7 = 0$
(c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$

26. The area of the circle and the area of a regular polygon inscribed the circle of n sides and of perimeter equal to that of the circle are in the ratio of
(a) $\tan\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$ (b) $\cos\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$
(c) $\sin\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$ (d) $\cot\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$

27. The ex-radii of a triangle r_1, r_2, r_3 are in H.P., then the sides a, b, c are
(a) in H.P. (b) in A.P.
(c) in G.P. (d) None of these

28. In any triangle ABC, $\Sigma \frac{\sin^2 A + \sin A + 1}{\sin A}$ is always greater than
(a) 9 (b) 3
(c) 27 (d) None of these

29. If twice the square of the diameter of a circle is equal to half the sum of the squares of the sides of inscribed triangle ABC, then $\sin^2 A + \sin^2 B + \sin^2 C$ is equal to
(a) 1 (b) 2
(c) 4 (d) 8

30. If in a triangle ABC, $2\frac{\cos A}{a} + \frac{\cos B}{b} + 2\frac{\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$ then the value of the angle A is
(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/2$ (d) $\pi/6$

31. If in a triangle $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, then the triangle is
(a) right angled (b) isosceles
(d) equilateral (d) None of these

32. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then angle A is
(a) $\pi/6$ (b) $\pi/4$
(c) $5\pi/12$ (d) $\pi/2$

33. In any ΔABC, the distance of the orthocentre from the vertices A, B, C are in the ratio
(a) $\sin A : \sin B : \sin C$
(b) $\cos A : \cos B : \cos C$
(c) $\tan A : \tan B : \tan C$
(d) None of these

34. In a ΔABC, I is the incentre. The ratio $IA : IB : IC$ is equal to
(a) $\text{cosec } A/2 : \text{cosec } B/2 : \text{cosec } C/2$
(b) $\sin A/2 : \sin B/2 : \sin C/2$
(c) $\sec A/2 : \sec B/2 : \sec C/2$
(d) None of these

35. If in a triangle, R and r are the circumradius and inradius respectively then the Hormonic mean of the exradii of the triangle is
(a) $3r$ (b) $2R$
(c) $R + r$ (d) None of these

36. In a ΔABC, $a = 2b$ and $|A - B| = \pi/3$. Then the $\angle C$ is
(a) $\pi/4$ (b) $\pi/3$
(c) $\pi/6$ (d) None of these

37. In a ΔABC, $\tan A \tan B \tan C = 9$. For such triangles, if $\tan^2 A + \tan^2 B + \tan^2 C = \lambda$ then
(a) $9 \cdot \sqrt[3]{3} < \lambda < 27$ (b) $\lambda \leq 27$
(c) $\lambda < 9 \cdot \sqrt[3]{3}$ (d) $\lambda \nless 27$

38. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, then the remaining two sides are
(a) 2, 3 (b) 1, 2
(c) 3, 4 (d) 2, 2

39. If in a ΔABC, $a^2 \cos^2 A = b^2 + c^2$ then
(a) $A < \frac{\pi}{4}$ (b) $\frac{\pi}{4} < A < \frac{\pi}{2}$
(c) $A > \frac{\pi}{2}$ (d) $A = \frac{\pi}{2}$

40. In a ΔABC, the tangent of half the difference of two angles is one third the tangent of half the sum of the two angles. The ratio of the sides opposite the angles are
(a) 2 : 3 (b) 1 : 3
(c) 2 : 1 (d) 3 : 4

41. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C and Δ, the area of the triangle, then $p_1^{-1} + p_2^{-1} - p_3^{-1}$ is equal to
(a) $\frac{s-a}{\Delta}$ (b) $\frac{s-b}{\Delta}$
(c) $\frac{s-c}{\Delta}$ (d) $\frac{s}{\Delta}$

42. If the median of ΔABC through A is perpendicular to AB, then
(a) $\tan A + \tan B = 0$
(b) $2 \tan A + \tan B = 0$
(c) $\tan A + 2 \tan B = 0$
(d) None of these

43. In a triangle ABC, $\cos A + \cos B + \cos C = 3/2$, then the triangle is
(a) isosceles (b) right angled
(c) equilateral (d) None of these

44. If $A_1A_2A_3 \ldots A_n$ be a regular polygon of n sides and $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then
(a) $n = 5$ (b) $n = 6$
(c) $n = 7$ (d) None of these

45. If p is the product of the sines of angles of a triangle, and q the product of their cosines, the tangents of the angle are roots of the equation
(a) $qx^3 - px^2 + (1+q)x - p = 0$
(b) $px^3 - qx^2 + (1+p)x - q = 0$
(c) $(1+q)x^3 - px^2 + qx - q = 0$
(d) None of these

46. In a ΔABC, $\tan A/2 = 5/6$ and $\tan C/2 = 2/5$ then
(a) a, c, b are in A.P.
(b) a, b, c are in A.P.
(c) b, a, c are in A.P.
(d) a, b, c are in G.P.

47. In a triangle, the line joining the circumcentre to the incentre is parallel to BC, then $\cos B + \cos C =$
(a) 3/2 (b) 1
(c) 3/4 (d) 1/2

48. If the angles of a triangle are in the ratio 1 : 2 : 3, the corresponding sides are in the ratio
(a) $2:3:1$ (b) $\sqrt{3}:2:1$
(c) $2:\sqrt{3}:1$ (d) $1:\sqrt{3}:2$

49. In a ΔABC, $a\cot A + b\cot B + c\cot C =$
(a) $r + R$ (b) $r - R$
(c) $2(r+R)$ (d) $2(r-R)$

50. In a triangle, the lengths of the two larger sides are 10 and 9. If the angles are in A.P., then the length of the third side can be
(a) $5 \pm \sqrt{6}$ (b) $3\sqrt{3}$
(c) 5 (d) $\sqrt{5} \pm 6$

51. In a triangle, $a^2 + b^2 + c^2 = ca + ab\sqrt{3}$. Then the triangle is
(a) equilateral
(b) right angled and isosceles
(c) right angled with $A = 90^\circ$, $B = 60^\circ$, $C = 30^\circ$
(d) None of the above

52. Three equal circles each of radius r touch one another. The radius of the circle touching all the three given circles internally is
(a) $(2+\sqrt{3})\,r$ (b) $\frac{(2+\sqrt{3})}{\sqrt{3}}\,r$
(c) $\frac{(2-\sqrt{3})}{\sqrt{3}}\,r$ (d) $(2-\sqrt{3})\,r$

53. In a triangle ABC; AD, BE and CF are the altitudes and R is the circum radius, then the radius of the circle DEF is
(a) $2R$ (b) R
(c) $R/2$ (d) None of these

54. A right angled trapezium is circumscribed about a circle. The radius of the circle. If the lengths of the bases (i.e., parallel sides) are equal to a and b is
(a) $a + b$ (b) $\frac{ab}{a+b}$
(c) $\frac{a+b}{ab}$ (d) $|a - b|$

55. If a, b, c, d are the sides of a quadrilateral, then the minimum value of $\frac{a^2+b^2+c^2}{d^2}$ is
(a) 1 (b) 1/2
(c) 1/3 (d) 1/4

56. If r_1, r_2, r_3 are the radii of the escribed circles of a triangle ABC and if r is the radius of its incircle, then $r_1r_2r_3 - r(r_1r_2 + r_2r_3 + r_3r_1)$ is equal to
(a) 0 (b) 1
(c) 2 (d) 3

57. If in ΔABC, $\frac{\sin A}{c} + \frac{\sin C}{a} = \frac{a}{bc} + \frac{c}{ab}$ then the trangle is
(a) right angled (b) isosceles
(c) equilateral (d) None of these

58. A triangle ABC exists such that
(a) $(b+c+a)(b+c-a) = 5bc$
(b) the sides are of length $\sqrt{19}, \sqrt{38}, \sqrt{116}$
(c) $\frac{b^2-c^2}{a^2} + \frac{c^2-a^2}{b^2} + \frac{a^2-b^2}{c^2} = 0$
(d) $\cos\left(\frac{B-C}{2}\right) = (\sin B + \sin C)\cos\left(\frac{B+C}{2}\right)$

59. The ratio of the areas of two regular octagons which are respectively inscribed and circumscribed to a circle of radius r is

(a) $\cos \frac{\pi}{8}$ (b) $\sin^2 \frac{\pi}{8}$

(c) $\cos^2 \frac{\pi}{8}$ (d) $\tan^2 \frac{\pi}{8}$

60. In a triangle ABC right angled at B, the inradius is

(a) $AB + BC - AC$

(b) $AB + AC - BC$

(c) $\frac{AB + BC - AC}{2}$

(d) None of these

MULTIPLE CHOICE – II

Each question, in this part has one or more than one correct answer (s). For each question, write the letter(s) a, b, c, d corresponding to the correct answer (s).

61. If in a triangle ABC, $\angle B = 60$ then

(a) $(a - b)^2 = c^2 - ab$

(b) $(b - c)^2 = a^2 - bc$

(c) $(c - a)^2 = b^2 - ac$

(d) $a^2 + b^2 + c^2 = 2b^2 + ac$

62. Given an isosceles triangle with equal sides of length b, base angle $\alpha < \pi/4$. r the radii and O, I the centres of the circumcircle and incircle, respectively. Then

(a) $R = \frac{1}{2} b \operatorname{cosec} \alpha$

(b) $\Delta = 2b^2 \sin 2\alpha$

(c) $r = \frac{b \sin 2\alpha}{2(1 + \cos \alpha)}$

(d) $OI = \left| \frac{b \cos (3\alpha/2)}{2 \sin \alpha \cos (\alpha/2)} \right|$

63. In ΔABC, $A = 15°$, $b = 10\sqrt{2}$ cm the value of 'a' for which these will be a unique triangle meeting these requirement is

(a) $10\sqrt{2}$ cm (b) 15 cm

(c) $5(\sqrt{3} + 1)$ cm (d) $5(\sqrt{3} - 1)$ cm

64. If ΔABC; $a = 5$, $b = 4$, $A = \frac{\pi}{2} + B$ for the value of angle C

(a) can not be evaluated

(b) $\tan^{-1}\left(\frac{9}{40}\right)$ (c) $\tan^{-1}\left(\frac{1}{40}\right)$

(d) $2\tan^{-1}\left(\frac{1}{9}\right)$ (e) None of these

65. If $\tan A$, $\tan B$ are the roots of the quadratic $abx^2 - c^2 x + ab = 0$, where a, b, c are the sides of a triangle, then

(a) $\tan A = a/b$ (b) $\tan B = b/a$

(c) $\cos C = 0$ (d) $\tan A + \tan B = \frac{c^2}{ab}$

(e) $\sin^2 A + \sin^2 B + \sin^2 C = 2$

66. There exists a triangle ABC satisfying

(a) $\tan A + \tan B + \tan C = 0$

(b) $\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{7}$

(c) $(a + b)^2 = c^2 + ab$ and $\sqrt{2}(\sin A + \cos A) = \sqrt{3}$

(d) $\sin A + \sin B = \frac{\sqrt{3} + 1}{2}$, $\cos A \cos B = \frac{\sqrt{3}}{4} = \sin A \sin B$

67. In a ΔABC, $2\cos\left(\frac{A - C}{2}\right) = \frac{a + c}{\sqrt{(a^2 + c^2 - ac)}}$.

Then

(a) $B = \pi/3$

(b) $B = C$

(c) A, B, C are in A.P.

(d) $B + C = A$

68. If in ΔABC,

$$\frac{a \cos A + b \cos B + c \cos C}{a \sin B + b \sin C + c \sin A} = \frac{a + b + c}{9R}$$

then the triangle ABC is

(a) isosceles (b) equilateral

(c) right angled (d) None of these

69. In a triangle ABC, AD is the altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2 - c^2}$ then $\angle B =$

(a) 53° (b) 113°

(c) 87° (d) None of these

70. If a, b and c are the sides of a triangle such that $b \cdot c = \lambda^2$, then the relation is a, λ and A is ....

(a) $C \geq 2\lambda \sin C/2$ (b) $b \geq 2\lambda \sin A/2$

(c) $a \geq 2\lambda \sin A/2$ (d) None of these

71. In a ΔABC, $\tan C < 0$. Then

(a) $\tan A \tan B < 1$

(b) $\tan A \tan B > 1$

(c) $\tan A + \tan B + \tan C < 0$

(d) $\tan A + \tan B + \tan C > 0$

72. If the sines of the angles A and B of a triangle ABC satisfy the equation $c^2x^2 - c(a+b)x + ab = 0$, then the triangle

(a) is acute-angled

(b) is right-angled

(c) is obtuse-angled

(d) satisfies $\sin A + \cos A = (a+b)/c$

73. In a ΔABC $\tan A$ and $\tan B$ satisfy the inequation $\sqrt{3}x^2 - 4x + \sqrt{3} < 0$ then

(a) $a^2 + b^2 + ab > c^2$ (b) $a^2 + b^2 - ab < c^2$

(c) $a^2 + b^2 > c^2$ (d) None of these

74. If A, B, C are angles of a triangle such that the angle A is obtuse, then $\tan B \tan C <$

(a) 0 (b) 1

(c) 2 (d) 3

75. If the sines of the angles of a triangle are in the ratios 3 : 5 : 7 their cotangent are in the ratio

(a) 2 : 3 : 7 (b) 33 : 65 : – 15

(c) 65 : 33 : – 15 (d) None of these

76. For a triangle ABC, which of the following is true ?

(a) $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$

(b) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2+b^2+c^2}{2abc}$

(c) $\dfrac{\sin A}{a} + \dfrac{\sin B}{b} + \dfrac{\sin C}{c} = \dfrac{3}{2R}$

(d) $\dfrac{\sin 2A}{a^2} = \dfrac{\sin 2B}{b^2} = \dfrac{\sin 2C}{c^2}$

77. Let $f(x+y) = f(x).f(y)$ for all x and y and $f(1) = 2$. If in a triangle ABC, $a = f(3), b = f(1) + f(3), c = f(2) + f(3)$, then $2A =$

(a) C (b) $2C$

(c) $3C$ (d) $4C$

78. If sides of a triangle ABC are in $A.P.$ and a is the least side, then $\cos A$ equals

(a) $\dfrac{3c-2b}{2c}$ (b) $\dfrac{4c-3b}{2c}$

(c) $\dfrac{4a-3b}{2c}$ (d) None of these

79. If the angles of a triangle are in the ratio 2 : 3 : 7, then the sides opposite these angles are in the ratio

(a) $\sqrt{2} : 2 : \sqrt{3}+1$ (b) $2 : \sqrt{2} : \sqrt{3}+1$

(c) $1 : \sqrt{2} : \dfrac{\sqrt{2}}{\sqrt{3}-1}$ (d) $\dfrac{1}{\sqrt{2}} : 1 : \dfrac{\sqrt{3}+1}{2}$

80. In a ΔABC, $b^2 + c^2 = 1999a^2$, then $\dfrac{\cot B + \cot C}{\cot A} =$

(a) $\dfrac{1}{999}$ (b) $\dfrac{1}{1999}$

(c) 999 (d) 1999

81. There exists a triangle ABC satisfying the conditions

(a) $b \sin A = a, A < \pi/2$

(b) $b \sin A > a, A > \pi/2$

(c) $b \sin A > a, A < \pi/2$

(d) $b \sin A < a, A < \pi/2, b > a$

82. If $\cos(\theta - \alpha), \cos\theta, \cos(\theta + \alpha)$ are in H.P., then $\cos\theta \sec \alpha/2$ is equal to

(a) –1 (b) $-\sqrt{2}$

(c) $\sqrt{2}$ (d) 2

83. If $\sin\beta$ is the G.M. between $\sin\alpha$ and $\cos\alpha$, then $\cos 2\beta$ is equal to

(a) $2\sin^2\left(\dfrac{\pi}{4} - \alpha\right)$ (b) $2\cos^2\left(\dfrac{\pi}{4} - \alpha\right)$

(c) $2\cos^2\left(\dfrac{\pi}{4} + \alpha\right)$ (d) $2\sin^2\left(\dfrac{\pi}{4} + \alpha\right)$

84. If l is the median from the vertex A to the side BC of a ΔABC, then

(a) $4l^2 = 2b^2 + 2c^2 - a^2$

(b) $4l^2 = b^2 + c^2 + 2bc \cos A$

(c) $4l^2 = a^2 + 4bc \cos A$

(d) $4l^2 = (2s - a)^2 - 4bc \sin^2 A/2$

85. If in a ΔABC, $r_1 = 2r_2 = 3r_3$, then

(a) $a/b = 4/5$

(b) $a/b = 5/4$

(c) $a/c = 3/5$

(d) $a/c = 5/3$

86. In a ΔABC, $2\cos A = \frac{\sin B}{\sin C}$ and $2^{\tan B}$ is a solution of equation $x^2 - 9x + 8 = 0$, then ΔABC is

(a) equilateral (b) isosceles
(c) scalene (d) right-angled

87. If $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then a^2, b^2, c^2 are in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

88. In ΔABC, $A : B : C = 3 : 5 : 4$. Then $a + b + c\sqrt{2}$ is equal to

(a) $2b$ (b) $2c$
(c) $3b$ (d) $3a$

89. If in an obtuse-angled triangle the obtuse angle is $3\pi/4$ and the other two angles are equal to two values of θ satisfying $a\tan\theta + b\sec\theta = c$, when $|b| \le \sqrt{(a^2 + c^2)}$, then $a^2 - c^2$ is equal to

(a) ac (b) $2ac$
(c) a/c (d) None of these

90. In ΔABC, $A = \pi/3$ and $b : c = 2 : 3$. If $\tan\theta = \frac{\sqrt{3}}{5}$, $0 < \theta < \pi/2$, then

(a) $B = 60° + \theta$ (b) $C = 60° + \theta$
(c) $B = 60° - \theta$ (d) $C = 60° - \theta$

Practice Test

M.M. : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. If A, A_1, A_2, A_3 are the areas of incircle and the ex-circles of a triangle, then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} =$

(a) $\frac{2}{\sqrt{A}}$ (b) $\frac{1}{\sqrt{A}}$
(c) $\frac{1}{2\sqrt{A}}$ (d) $\frac{3}{\sqrt{A}}$

2. In any ΔABC

$\Pi\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)$ is always greater than

(a) 9 (b) 3
(c) 27 (d) None of these

3. If A is the area and $2s$ the sum of three sides of a triangle, then

(a) $A < \frac{s^2}{3\sqrt{3}}$ (b) $a < \frac{s^2}{2}$
(c) $A > \frac{s^2}{\sqrt{3}}$ (d) None of these

4. In ΔABC, which of the following statements are true

(a) maximum value of $\sin 2A + \sin 2B + \sin 2C$ is same as the maximum value of $\sin A + \sin B + \sin C$

(b) $R \ge 2r$ where R is circumradius and r is inradius

(c) $R^2 > \frac{abc}{a+b+c}$

(d) ΔABC is right angled if $r + 2R = s$ where s is semi perimeter

(e) None of these

5. In a triangle if $r_1 > r_2 > r_3$, then

(a) $a > b > c$ (b) $a < b < c$
(c) $a > b$ & $b < c$ (d) $a < b$ & $b > c$

6. If there are only two linear functions f and g which map [1, 2] on [4, 6] and in a ΔABC, $c = f(1) + g(1)$ and a is the maximum value of r^2, where r is the distance of a variable point on the curve $x^2 + y^2 - xy = 10$ from the origin, then $\sin A : \sin C$ is

(a) 1 : 2 (b) 2 : 1
(c) 1 : 1 (d) None of these

7. In a triangle $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$

(a) $\cot A, \cot B, \cot C$ in A.P.
(b) $\sin 2A, \sin 2B, \sin 2C$ in A.P.
(c) $\cos 2A, \cos 2B, \cos 2C$ in A.P.
(d) $a\sin A$, $b\sin B$, $c\sin C$ in A.P.

8. If in a ΔABC $\sin C + \cos C + \sin(2B + C) - \cos(2B + C) = 2\sqrt{2}$, then ΔABC is

(a) equilateral (b) isosceles
(c) right angled (d) obtuse angled

9. The radius of the circle passing through the centre of incircle of ΔABC and through the end points of BC is given by
(a) $(a/2)\ \cos A$ (b) $(a/2)\ \sec A/2$
(c) $(a/2)\ \sin A$ (d) $a \sec A/2$

10. In a triangle ABC, $\sqrt{a} + \sqrt{b} - \sqrt{c}$ is
(a) always positive
(b) always negative
(c) positive only when c is smallest
(d) None of these

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple choice–I

1. (c) **2.** (c) **3.** (c) **4.** (a) **5.** (a) **6.** (d)
7. (a) **8.** (a) **9.** (b) **10.** (c) **11.** (b) **12.** (a)
13. (a) **14.** (a) **15.** (a) **16.** (b) **17.** (a) **18.** (c)
19. (c) **20.** (c) **21.** (a) **22.** (c) **23.** (c) **24.** (a)
25. (a) **26.** (a) **27.** (b) **28.** (a) **29.** (c) **30.** (c)
31. (a) **32.** (c) **33.** (b) **34.** (a) **35.** (a) **36.** (b)
37. (b) **38.** (a) **39.** (c) **40.** (c) **41.** (c) **42.** (c)
43. (c) **44.** (c) **45.** (a) **46.** (a) **47.** (b) **48.** (d)
49. (c) **50.** (a) **51.** (c) **52.** (b) **53.** (c) **54.** (b)
55. (c) **56.** (a) **57.** (a) **58.** (d) **59.** (c) **60.** (c)

Multiple Choice–II

61. (c), (d) **62.** (a). (c), (d) **63.** (a), (d) **64.** (b), (d) **65.** (a), (b), (c), (d), (e)
66. (c), (d) **67.** (a), (c) **68.** (b) **69.** (b) **70.** (c) **71.** (a), (c)
72. (b), (d) **73.** (a), (b) **74.** (b) **75.** (c) **76.** (b), (c) **77.** (a)
78. (b) **79.** (a), (c), (d) **80.** (a) **81.** (a,) (d) **82.** (b), (c) **83.** (a), (c)
84. (a), (b), (c), (d) **85.** (b), (d) **86.** (a) **87.** (a) **88.** (c)
89. (b) **90.** (b)

Practice Test

1. (b) **2.** (c) **3.** (a) **4.** (a), (b), (c), (d) **5.** (a)
6. (c) **7.** (a), (c), (d) **8.** (b), (c) **9.** (b) **10.** (a)

31

HEIGHTS AND DISTANCES

§ 31.1. If O be the observer's eye and OX be the horizontal line through O. If the object P is at a higher level than O, then angle $POX = \theta$ is called the **angle of elevation.**

If the object P is at a lower than O, then angle POX is called the **angle of depression.**

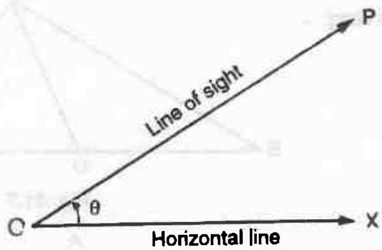


Fig. 31.1.

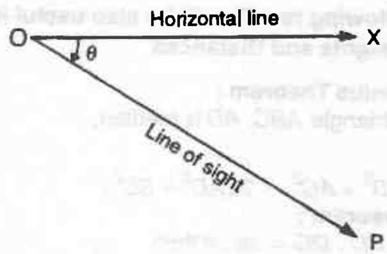


Fig. 31.2.

§ 31.2. Properties of Circle

(i) Angles of the same segment of a circle are equal
i.e., $\angle APB = \angle AQB = \angle ARB$

(ii) Angles of alternate segment of a circle are equal

(iii) If the line joining two points A and B subtend the greatest angle α at a point P then the circle, will touch the straight line, at the point P.

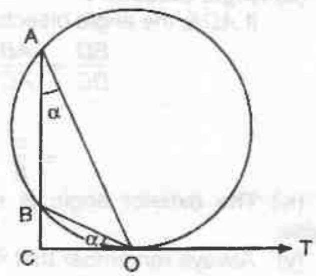


Fig. 31.3.

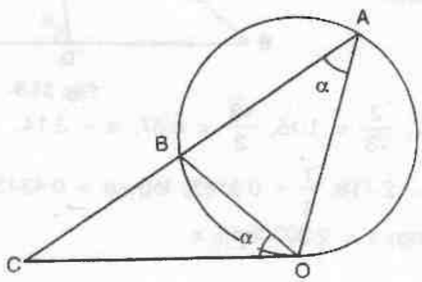


Fig. 31.4.

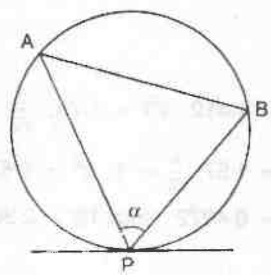


Fig. 31.5.

(iv) The angle subtended by any chord on centre is twice the angle subtended by the same chord on any point on the circumference of the circle.

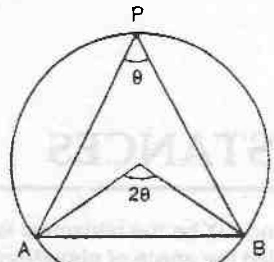


Fig. 31.6.

§ 31.3. The following results will be also useful in solving problems of Heights and Distances

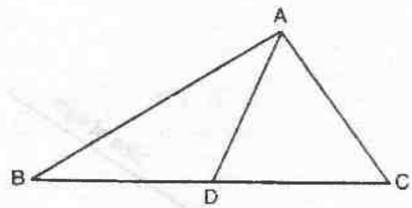


Fig. 31.7.

(i) **Appollonius Theorem :**

If in a triangle ABC, AD is median,

then

$$AB^2 + AC^2 = 2\,(AD^2 + BD^2)$$

(ii) ***m–n* Theorem :**

If $BD : DC = m : n$ then

$$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$$

or $$(m + n)\cot\theta = n\cot B - m\cot C.$$

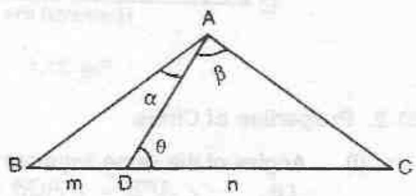


Fig. 31.8.

(iii) **Angle Bisector :**

If AD is the angle bisector of $\angle BAC$

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC}$$

$$= \frac{c}{b}$$

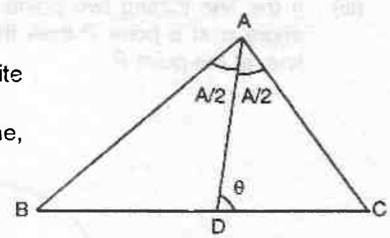


Fig. 31.9.

(iv) The exterior angle is equal to sum of interior opposite angles.

(v) Always remember that if a line is perpendicular to a plane, then its perpendicular to every line in that plane.

Remember :

$\sqrt{2} = 1.412$, $\sqrt{3} = 1.73$, $\frac{1}{\sqrt{2}} = 0.7$, $\frac{2}{\sqrt{3}} = 1.15$, $\frac{\sqrt{3}}{2} = 0.87$, $\pi = 3.14$,

$\frac{\pi}{2} = 1.57$, $\frac{\pi}{3} = 1$, $\pi^2 = 9.87$, $e = 2.718$, $\frac{1}{\pi} = 0.3183$, $\log_{10} e = 0.4343$,

$\log_e \pi = 0.4972$; $\log_e 10 = 2.303$; $\log_e x = 2.303 \log_{10} x$.

MULTIPLE CHOICE –I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. An isosceles triangle of wood is placed in a vertical plane, vertex upwards and faces the sun. If $2a$ be the base of the triangle, h its height and 30° the altitude of the sun, then the tangent of the angle at the apex of the shadow is
(a) $\frac{2ah\sqrt{3}}{3h^2 - a^2}$ (b) $\frac{2ah\sqrt{3}}{3h^2 + a^2}$
(c) $\frac{ah\sqrt{3}}{h^2 - a^2}$ (d) None of these

2. As seen from A, due west of a hill HL itself leaning east, the angle of elevation of top H of the hill is 60°; and after walking a distance of one kilometer along an incline of 30° to a point B, it was seen that the hill LH was printed at right angles to AB. The height LH of the hill is
(a) $\frac{1}{\sqrt{3}}$ km (b) $\sqrt{3}$ km
(c) $2\sqrt{3}$ km (d) $\frac{2}{\sqrt{3}}$ km

3. A tower subtends angles θ, 2θ and 3θ at 3 points A, B, C respectively, lying on a horizontal line through the foot of the tower then the ratio AB/BC equals to
(a) $\frac{\sin 3\theta}{\sin\theta}$ (b) $\frac{\sin\theta}{\sin 3\theta}$
(c) $\frac{\cos 3\theta}{\cos\theta}$ (d) $\frac{\tan\theta}{\tan 3\theta}$

4. ABC is a triangular park with $AB = AC = 100$ metres. A clock tower is situated at the mid point of BC. The angles of elevation of the top of the tower at A and B are $\cot^{-1} 3.2$ and $\operatorname{cosec}^{-1} 2.6$ respectively. The height of the tower is
(a) 16 mt (b) 25 mt
(c) 50 mt (d) None of these

5. From a station A due west of a tower the angle of elevation of the top of the tower is seen to be 45°. From a station B, 10 metres from A and in the direction 45° south of east the angle of elevation is 30°, the height of the tower is
(a) $5\sqrt{2}(\sqrt{5}+1)$ metres
(b) $\frac{5\sqrt{2}(\sqrt{5}+1)}{2}$ metres
(c) $\frac{5(\sqrt{5}+1)}{2}$ metres
(d) None of these

6. A tree is broken by wind, its upper part touches the ground at a point 10 metres from the foot of the tree and makes an angle of 45° with the ground. The entire length of the tree is
(a) 15 metres
(b) 20 metres
(c) $10(1+\sqrt{2})$ metres
(d) $10\left(1+\frac{\sqrt{3}}{2}\right)$ metres

7. A person towards a house observes that a flagstaff on the top subtends the greatest angle θ when he is at a distance d from the house. The length of the flagstaff is
(a) $\frac{3}{2}d\tan\theta$ (b) $d\cot\theta$
(c) $2d\tan\theta$ (d) None of these

8. A tower and a flag staff on its top subtend equal angles at the observer's eye. if the heights of flagstaff, tower and the eye of the observer are respectively a, b and h. then the distance of the observer's eye from the base of the tower is
(a) $a\sqrt{\frac{a+b-2h}{a+b}}$
(b) $b\sqrt{\left(\frac{a+b-2h}{a-b}\right)}$
(c) $a\sqrt{\left(\frac{a-h}{a+b}\right)}$
(d) None of these

9. The angle of elevation of the top of a tower from a point A due south of it, is $\tan^{-1} 6$ and that from B due east of it, is $\tan^{-1} 7.5$. If h is

the height of the tower, then $AB = \lambda h$, where $\lambda^2 =$

(a) 21/700 (b) 42/1300
(c) 41/900 (d) None of these

10. A ladder rests against a wall at an angle α to the horizontal. If foot is pulled away through a distance a, so that it slides a distance b down the wall, finally making an angle β with the horizontal. Then

$$\tan\left(\frac{\alpha+\beta}{2}\right) =$$

(a) b/a (b) a/b
(c) $a-b$ (d) $a+b$

11. Two rays are drawn through a point A at an angle of 30°. A point B is taken on one of them at distance a from the point A. A perpendicular is drawn from the point B to the other ray, another perpendicular is drawn from its foot to AB, and so on, then the length of the resulting infinite polygonal line is

(a) $2\sqrt{3}\,a$ (b) $a(2+\sqrt{3})$
(c) $a(2-\sqrt{3})$ (d) None of these

12. From the top of a cliff h metres above sea level an observer notices that the angles of depression of an object A and its image B are complementry. if the angle of depression at A is θ, The height of A above sea level is

(a) $h\sin\theta$ (b) $h\cos\theta$
(c) $h\sin 2\theta$ (d) $h\cos 2\theta$

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer(s). For each question, write the letters(s) a, b, c, d corresponding to the correct answer(s) :

13. A person standing at the foot of a tower walks a distance $3a$ away from the tower and observes that the angle of elevation of the top of the tower is α. He then walks a distance $4a$ perpendicular to the previous direction and observes the angle of elevation to be β. The height of the tower is

(a) $3a\tan\alpha$ (b) $5a\tan\beta$
(c) $4a\tan\beta$ (d) $7a\tan\beta$

14. A tower subtends an angle of 30° at a point on the same level as the foot of the tower. At a second point, h meter above the first, the depression of the foot of the tower is 60°, horizontal distance of the tower from the point is

(a) $h\cot 60°$ (b) $\frac{1}{3}h\cot 30°$
(c) $\frac{1}{3}h\cot 60°$ (d) $h\cot 30°$

15. The upper (3/4)th portion of a vertical pole subtends an angle $\tan^{-1}(3/5)$ at a point in horizontal plane through its foot and distant 40 m from it. The height of the pole is

(a) 40 m (b) 160 m
(c) 10 m (d) 200 m

16. The angles of elevation of the top of a tower from two points at a distance of 49 metre and 64 metre from the base and in the same straight line with it are complementry the distances of the points from the top of the tower are

(a) 74·41 (b) 74·28
(c) 85·04 (d) 84·927

17. The angle of elevation of the top P of a pole OP at a point A on the ground is 60°. There is a mark on the pole at Q and the angle of elevation of this mark at A is 30°. Then if $PQ = 400$ cm

(a) $OA = 346{\cdot}4$ cm
(b) $OP = 600$ cm
(c) $AQ = 400$ cm
(d) $AP = 946{\cdot}4$ cm

18. $ABCD$ is a square plot. The angle of elevation of the top of a pole standing at D from A or C is 30° and that from B is θ then $\tan\theta$ is equal to

(a) $\sqrt{6}$ (b) $1/\sqrt{6}$
(c) $\sqrt{3}/\sqrt{2}$ (d) $\sqrt{2}/\sqrt{3}$

19. If a flagstaff subtends the same angle at the points A, B, C, D on the horizontal plane through the foot of the flagstaff then A, B, C, D are the vertices of a

(a) square (b) cyclic quadrilateral

(c) rectangel (d) None of these

20. The angle of elevation of the top of a T.V. tower from three points A, B, C in a straight line, (in the horizontal plane) through the foot of the tower are α, 2α, 3α respectively. If $AB = a$, the height of the tower is

(a) $a \tan \alpha$ (b) $a \sin \alpha$

(c) $a \sin 2\alpha$ (d) $a \sin 3\alpha$

Practice Test

M.M. : 10 Time : 15 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. From the top of a building of height h, a tower standing on the ground is observed to make an angle θ. If the horizontal distance between the building and the tower is h, the height of the tower is

(a) $\dfrac{2h \sin \theta}{\sin \theta + \cos \theta}$

(b) $\dfrac{2h \tan \theta}{1 + \tan \theta}$

(c) $\dfrac{2h}{1 + \cot \theta}$

(d) $\dfrac{2h \cos \theta}{\sin \theta + \cos \theta}$

2. From the top of a light house, the angle of depression of two stations on opposite sides of it at a distance a apart are α and β. The height of the light house is

(a) $\dfrac{a \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}$ (b) $\dfrac{a \cot \alpha \cot \beta}{\cot \alpha + \cot \beta}$

(c) $\dfrac{a}{\cot \alpha + \cot \beta}$ (d) $\dfrac{a}{\cot \alpha \cot \beta}$

3. A vertical lamp-post, $6m$ high, stands at a distance of 2m from $2m$ from a wall, $4m$ high. A $1{\cdot}5m$ tall man starts to walk away from the wall on the other side of the wall, in line with the lamp-post. The maximum distance to which the man can walk remaining in the shadow is

(a) $\dfrac{5}{2}m$ (b) $\dfrac{3}{2}m$

(c) $4m$ (d) None of these

4. The angle of elevation of the top of a tower standing on a horizontal plane, from two points on a line passing through its foot at distances a and b respectively, are complementary angles. If the line joining the two points subtends an angle θ at the top of the tower, then

(a) $\sin \theta = \dfrac{a - b}{a + b}$

(b) $\tan \theta = \dfrac{2\sqrt{ab}}{a - b}$

(c) $\sin \theta = \dfrac{a + b}{a - b}$

(d) $\cot \theta = \dfrac{2\sqrt{ab}}{a - b}$

5. A man standing between two vertical posts finds that the angle subtended at his eyes by the tops of the posts is a right angle. If the heights of the two posts are two times and four times the height of the man, and the distance between them is euqal to the length of the longer post, then the ratio of the distances of the man from the shorter and the longer post is

(a) 3 : 1 (b) 3 : 2

(c) 1 : 3 (d) 2 : 3

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice – I

1. (a) **2.** (d) **3.** (a) **4.** (b) **5.** (d) **6.** (c)
7. (c) **8.** (b) **9.** (c) **10.** (b) **11.** (b) **12.** (d)

Multiple Choice – II

13. (a), (b) **14.** (a), (b) **15.** (a), (b) **16.** (a), (c) **17.** (a), (b), (c) **18.** (b)
19. (b) **20.** (c)

Practice Test

1. (a), (b), (c) **2.** (a), (c) **3.** (a) **4.** (a), (d) **5.** (a), (c).

32

VECTORS

1. Linearly Independent Vectors :

A set of vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ is said to be linearly independent iff

$$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = 0 \Rightarrow x_1 = x_2 = \ldots = x_n = 0$$

2. Linearly Dependent Vectors :

A set of vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ is said to be linearly dependent iff there exists scalars $x_1, x_2, \ldots, x_n$ not all zero such that

$$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = 0.$$

3. Test of Collinearity :

(i) Two vectors $\vec{a}$ and $\vec{b}$ are collinear $\Leftrightarrow \vec{a} = \lambda\vec{b}$ for some scalar λ.

(ii) Three vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear, if there exists scalars x, y, z such that

$$x\vec{a} + y\vec{b} + z\vec{c} = 0, \text{ where } x + y + z - 0;$$

Also the points A, B, C are collinear if $\vec{AB} = \lambda\,\vec{BC}$ for some scalar λ.

4. Test of Coplanarity :

(i) Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if one of them is a linear combination of the other two if there exist scalars x and y such that $\vec{c} = x\vec{a} + y\vec{b}$

(ii) Four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar if $\exists$ scalars x, y, z, w not all zero simultaneously such that

$$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0 \text{ where } x + y + z + w = 0.$$

5. Scalar or Dot product :

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta \qquad (0 < \theta < \pi)$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

Properties of the Scalar product :

(i) $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = a^2$

(ii) Two vectors $\vec{a}$ and $\vec{b}$ make an acute angle if $\vec{a} \cdot \vec{b} > 0$, an obtuse angle if $\vec{a} \cdot \vec{b} < 0$ and are inclined at a right angle if $\vec{a} \cdot \vec{b} = 0$.

(iii) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

(iv) Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

(v) Components of $\vec{r}$ in the direction of $\vec{a}$ and perpendicular to $\vec{a}$ are

$$\left[\frac{\vec{r} \cdot \vec{a}}{|\vec{a}|^2}\right]\vec{a} \text{ and } \vec{r} - \left[\frac{\vec{r} \cdot \vec{a}}{|\vec{a}|^2}\right]\vec{a} \text{ respectively.}$$

(vi) If $\hat{i}$, $\hat{j}$ and $\hat{k}$ are three unit vectors along three mutually perpendicular lines, then

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 \text{ and } \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$$

(vii) **Work done :** If a force $\vec{F}$ acts at point A and displaces it to the point B, then the work done by the force $\vec{F}$ is $\vec{F}.\vec{AB}$.

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\text{and } |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2},$$

$$|\vec{b}| = \sqrt{(b_1^2 + b_2^2 + b_3^2)}$$

if angle between $\vec{a}$ and $\vec{b}$ is θ then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}}$$

(ix) $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = a^2 - b^2$

$$(\vec{a} + \vec{b})^2 = a^2 + b^2 + 2\vec{a}.\vec{b}$$

6. Vector or Cross Product :

The vector product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, $(0 < \theta < \pi)$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $\hat{n}$ is the unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

Properties of the Vector Product :

(i) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (*i.e.*, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$)

(ii) $\vec{a} \times \vec{a} = 0$

(iii) $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a}.\vec{b})^2$

(iv) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(v) The vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by $\vec{a} \times \vec{b}$.

(vi) The unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is

$$\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

and a vector of magnitude λ perpendicular to the plane of ($\vec{a}$ and $\vec{b}$ or $\vec{b}$ and $\vec{a}$) is

$$\pm \frac{\lambda(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}.$$

(vii) If $\hat{i}$, $\hat{j}$, $\hat{k}$ are three unit vectors along three mutually perpendicular lines, then

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \text{ \&}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}.$$

(viii) If $\vec{a}$ and $\vec{b}$ are collinear then

$$\vec{a} \times \vec{b} = 0.\ (\vec{a}, \vec{b} \neq 0)$$

(ix) **Moment :** The moment of a force $\vec{F}$ applied at A about the point B is the vector $\vec{BA} \times \vec{F}$.

(x) (a) The area of a triangle if adjacent sides are $\vec{a}$ and $\vec{b}$ is given by

$$\frac{1}{2}|\vec{a} \times \vec{b}|$$

(b) The area of a parallelogram if adjacent sides are $\vec{a}$ and $\vec{b}$ is given by

$$|\vec{a} \times \vec{b}|$$

(c) The area of a parallelogram if diagonals are $\vec{c}$ and $\vec{d}$ is given by

$$= \frac{1}{2} |\vec{c} \times \vec{d}|$$

7. Scalar Triple Product :

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, there $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is called the scalar triple product of these three vectors.

Note : The scalar triple product is usually written as $(\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$ or $[\vec{a}, \vec{b}, \vec{c}]$

Properties of scalar Triple product :

(i) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

(ii) $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(iii) If λ is a scalar then

$$[\lambda \vec{a}, \vec{b}, \vec{c}] = \lambda [\vec{a}, \vec{b}, \vec{c}]$$

(iv) If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ and $\vec{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \times \vec{b} \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(v) The value of scalar triple product, if two of its vectors are equal, is zero.

i.e., $[\vec{a}\ \vec{b}\ \vec{b}] = 0$

(vi) $[\vec{a}\ \vec{b}\ \vec{c_1} + \vec{c_2}] = [\vec{a}\ \vec{b}\ \vec{c_1}] + [\vec{a}\ \vec{b}\ \vec{c_2}]$

(vii) The volume of the parallelopiped whose adjacent sides are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $[\vec{a}\ \vec{b}\ \vec{c}]$

(viii) The volume of the tetrahedran whose adjacent sides are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

(ix) The volume of the triangular prism whose adjacent sides are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\frac{1}{2}[\vec{a}\ \vec{b}\ \vec{c}]$.

(x) If $[\vec{a}\ \vec{b}\ \vec{c}] = 0 \Leftrightarrow \vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

(xi) If $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{d}\ \vec{a}\ \vec{b}] + [\vec{d}\ \vec{b}\ \vec{c}] + [\vec{d}\ \vec{c}\ \vec{a}] \Leftrightarrow \vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar.

(xii) Three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed or left handed system according as

$$[\vec{a}\ \vec{b}\ \vec{c}] > \text{ or } < 0.$$

(xiii) $$[\vec{a}\ \vec{b}\ \vec{c}]\ [\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} \vec{a} \cdot \vec{u} & \vec{b} \cdot \vec{u} & \vec{c} \cdot \vec{u} \\ \vec{a} \cdot \vec{v} & \vec{b} \cdot \vec{v} & \vec{c} \cdot \vec{v} \\ \vec{a} \cdot \vec{w} & \vec{b} \cdot \vec{w} & \vec{c} \cdot \vec{w} \end{vmatrix}$$

(xiv) $[\vec{a}, \vec{b} + \vec{d}, \vec{c} + \vec{r}] = [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{r}] + [\vec{a}\ \vec{d}\ \vec{c}] + [\vec{a}\ \vec{d}\ \vec{r}]$

8. Vector Triple Product

The vector triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c})$.

and $$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Also $$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

clearly $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$; in general.

Equality holds if either $\vec{a}$ or $\vec{b}$ or $\vec{c}$ is zero or $\vec{c}$ is collinear with $\vec{a}$ or $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c}$.

9. Scalar product of four Vectors

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, the products $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is called scalar products of four vectors.

$$i.e., (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$$

This relation is known as **Lagrange's Identity.**

10. Vector product of four Vectors

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, the products $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is called vector products of four vectors.

i.e., $$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Also, $$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\ \vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\ \vec{a}.$$

An expression for any vector $\vec{r}$, in space, as a linear combination of three non coplanar vectors $\vec{a}, \vec{b}, \vec{c}$.

$$\vec{r} = \frac{[\vec{r}\ \vec{b}\ \vec{c}]\ \vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\ \vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\ \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

11. Reciprocal System of Vectors

If $\vec{a}, \vec{b}, \vec{c}$ be any three non-coplanar vectors so that $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$, then the three vectors $\vec{a}', \vec{b}', \vec{c}'$ defined by the equations

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \quad \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \quad \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

are called reciprocal system of vectors to the vectors $\vec{a}, \vec{b}, \vec{c}$.

Properties of Reciprocal System of Vectors

(i) $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

(ii) $\vec{a}.\vec{b}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = 0$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a}'\ \vec{b}'\ \vec{c}'] = 1$

(iv) $\vec{a} = \dfrac{\vec{b}' \times \vec{c}'}{[\vec{a}', \vec{b}', \vec{c}']}, \quad \vec{b} = \dfrac{\vec{c}' \times \vec{a}'}{[\vec{a}', \vec{b}', \vec{c}']}, \quad \vec{c} = \dfrac{\vec{a}' \times \vec{b}'}{[\vec{a}', \vec{b}', \vec{c}']}$

(v) The system of unit vectors $\hat{i}, \hat{j}, \hat{k}$ is its own reciprocal.
i.e., $\hat{i}' = \hat{i}, \hat{j}' = \hat{j}$ and $\hat{k}' = \hat{k}$.

APPLICATION OF VECTORS TO GEOMETRY

(1) Bisector of an angle : The bisectors of the angles between the lines $\vec{r} = x\vec{a}$ and $\vec{r} = y\vec{b}$ are given by

$$\vec{r} = \lambda \left(\frac{\vec{a}}{|\vec{a}|} \pm \frac{\vec{b}}{|\vec{b}|} \right) \quad (\lambda \in R)$$

'+' sign gives internal bisector and '—' sign gives external bisector.

(2) Section formula : If $\vec{a}$ and $\vec{b}$ are the P.V. of A and B and $\vec{r}$ be the P.V. of the point P which divides to join of A and B in the ratio $m : n$ then

$$\vec{r} = \frac{m\vec{b} \pm n\vec{a}}{m \pm n}$$

'+' sign takes for internal
'—' sign takes for external.

(3) If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the P.V. of three vertices of $\Delta\, ABC$ and $\vec{r}$ be the P.V. of the centroid of $\Delta\, ABC$, then

$$\vec{r} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

(4) Equation of straight line

(i) Vector equation of the straight line passing through origin and parallel to $\vec{b}$ is given by $\vec{r} = t\,\vec{b}$, where t is scalar.

(ii) Vector equation of the straight line passing through $\vec{a}$ and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + t\,\vec{b}$, where t is scalar.

(iii) Vector equation of the straight line passing through $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + t\,(\vec{b} - \vec{a})$ when t is scalar.

(5) Equation of a plane

(i) Vector equation of the plane through origin and parallel to $\vec{b}$ and $\vec{c}$ is given by

$$\vec{r} = s\,\vec{b} + t\,\vec{c}$$

where s and t are scalars.

(ii) Vector equation of the plane passing through $\vec{a}$ and paralel to $\vec{b}$ and $\vec{c}$ is given by $\vec{r} = \vec{a} + s\,\vec{b} + t\,\vec{c}$ where s and t are scalars.

(iii) Vector equation of the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\vec{r} = (1 - s - t)\,\vec{a} + s\,\vec{b} + t\,\vec{c}$$

where s and t are scalars.

(6) Perpendicular distance of the line

$\vec{r} = \vec{a} + t\,\vec{b}$ from the point P (*P.V.* $\vec{c}$)

$$= \frac{|(\vec{c} - \vec{a}) \times \vec{b}|}{|\vec{b}|}$$

(7) Perpendicular distance of the plane *i.e.*, $\vec{r} \cdot \vec{n} = p$ from the point P (*P.V.* $\vec{a}$)

$$= \frac{|\vec{a} \cdot \vec{n} - p|}{|\vec{n}|}$$

(8) The condition that two lines $\vec{r} = \vec{a} + t\,\vec{b}$, and $\vec{r} = \vec{c} + t_1\,\vec{d}$ (where t and t_1 are scalars) are coplaner (they much intersect) is given by

$$[\vec{a} - \vec{c},\ \vec{b},\ \vec{d}] = 0$$

(9) The shortest distance between two non intersecting lines (skew lines) $\vec{r} = \vec{a} + t\,\vec{b}$, and $\vec{r} = \vec{c} + t_1\,\vec{d}$ (where t and t_1 are scalars) is given by

$$= \frac{[\vec{b},\ \vec{d},\ (\vec{a} - \vec{c})]}{|\vec{b} \times \vec{d}|}$$

(10) Vector equation of sphere with centre $\vec{a}$ and radius p is

$$|\vec{r} - \vec{a}| = p$$

(11) Vector equation of sphere when extremities of diameter being $\vec{a}$, $\vec{b}$ is given by

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$$

MULTIPLE CHOICE – I

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. The two vectors $\{\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \vec{b} = 4\hat{i} - \lambda\hat{j} + 6\hat{k}\}$ are parallel if $\lambda =$
(a) 2 (b) –3
(c) 3 (d) –2

2. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then
(a) $\vec{a}$ is parallel to $\vec{b}$ (b) $\vec{a} \perp \vec{b}$
(c) $|\vec{a}| = |\vec{b}|$ (d) None of these

3. If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is the angle between them, then $\left|\frac{\vec{a} - \vec{b}}{2}\right|$ is
(a) $\sin\frac{\theta}{2}$ (b) $\sin\theta$
(c) $2\sin\theta$ (d) $\sin 2\theta$

4. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then the angle between $\vec{a}$ and $\vec{b}$ is
(a) $\pi/6$ (b) $\pi/3$
(c) $2\pi/3$ (d) $5\pi/3$

5. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, then the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$ is
(a) 1 (b) 3
(c) –3/2 (d) None of these

6. Vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle $\theta = 120°$. If $|\vec{a}| = 1, |\vec{b}| = 2$, then $\{(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})\}^2$ is equal to
(a) 225 (b) 275
(c) 325 (d) 300

7. If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a}\cdot\vec{b} = 0$ and $\vec{a} + \vec{b} = \vec{c}$ then
(a) $|\vec{a}|^2 + |\vec{b}|^2 = |\vec{c}|^2$
(b) $|\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2$
(c) $|\vec{b}|^2 = |\vec{a}|^2 + |\vec{c}|^2$
(d) None of these

8. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplaner vectors, then $[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}]$ is equal to
(a) 0 (b) $[\vec{a}, \vec{b}, \vec{c}]$
(c) $[\vec{a}, \vec{b}, \vec{c}]^2$ (d) $2[\vec{a}, \vec{b}, \vec{c}]$

9. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then $\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{b}\cdot(\vec{c}\times\vec{a})} + \frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\vec{c}\cdot(\vec{a}\times\vec{b})} + \frac{\vec{c}\cdot(\vec{b}\times\vec{a})}{\vec{a}\cdot(\vec{b}\times\vec{c})}$ is equal to
(a) 0 (b) 1
(c) 2 (d) None of these

10. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar unit vectors, then $[\vec{a}\ \vec{b}\ \vec{c}]$ is
(a) ± 1 (b) 0
(c) ± 3 (d) 2

11. The value of c so that for all real x, the vectors $cx\hat{i} - 6\hat{j} + 3\hat{k}$, $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle are
(a) $c < 0$ (b) $0 < c < 4/3$
(c) $-4/3 < c < 0$ (d) $c > 0$

12. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are three unit vectors, such that $\hat{a} + \hat{b} + \hat{c}$ is also a unit vectors θ_1, θ_2 and θ_3 are angles between the vectors $\hat{a}, \hat{b}$; $\hat{b}, \hat{c}$ and $\hat{c}, \hat{a}$ respectively, then among θ_1, θ_2 and θ_3
(a) all are acute angles
(b) all are right angles
(c) at least one is obtuse angle
(d) none of these

13. If $\hat{i}, \hat{j}, \hat{k}$ are unit orthonormal vectors and $\vec{a}$ is a vector if $\vec{a} \times \vec{r} = \hat{j}$, then $\vec{a}\cdot\vec{r}$ is
(a) –1 (b) 0
(c) 1 (d) None of these

14. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{a} + \vec{b}| = 5$, then $|\vec{a} - \vec{b}| =$
(a) 3 (b) 4
(c) 5 (d) 6

15. If $(\vec{a} \times \vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 144$ and $|\vec{a}| = 4$, then $|\vec{b}| =$
(a) 16 (b) 8
(c) 3 (d) 12

16. The projection of the vector $2\hat{i} + 3\hat{j} - 2\hat{k}$ on the vector $\hat{i} + 2\hat{j} + 3\hat{k}$ is
(a) $\frac{1}{\sqrt{14}}$ (b) $\frac{2}{\sqrt{14}}$
(c) $\frac{3}{\sqrt{14}}$ (d) None of these

17. For non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ $|(\vec{a}\times\vec{b}).\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$ holds iff
(a) $\vec{a}.\vec{b}=0, \vec{b}.\vec{c}=0$ (b) $\vec{b}.\vec{c}=0, \vec{c}.\vec{a}=0$
(c) $\vec{c}.\vec{a}=0, \vec{a}.\vec{b}=0$ (d) $\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$

18. If $\vec{a}$ is a unit vector such that $\vec{a}\times(\hat{i}+\hat{j}+\hat{k})=\hat{i}-\hat{k}$, then $\vec{a}=$
(a) $-\frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k})$
(b) $\hat{j}$
(c) $\frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k})$
(d) $\hat{i}$

19. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and α be the angle between them, then $\vec{a}+\vec{b}$ is a unit vector if
(a) $\alpha=\pi/4$ (b) $\alpha=\pi/3$
(c) $\alpha=2\pi/3$ (d) $\alpha=\pi/2$

20. If $\vec{a}+2\vec{b}+3\vec{c}=0$ and $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$ is equal to $\lambda(\vec{b}\times\vec{c})$ then $\lambda=$
(a) 3 (b) 4
(c) 5 (d) None of these

21. If $\vec{OA}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{OB}=3\hat{i}+\hat{j}-2\hat{k}$, then $\vec{OC}$ which bisects the angle AOB is given by
(a) $2\hat{i}-2\hat{j}-2\hat{k}$
(b) $2\hat{i}+2\hat{j}+2\hat{k}$
(c) $-2\hat{i}+2\hat{j}-2\hat{k}$
(d) $2\hat{i}+2\hat{j}-2\hat{k}$

22. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$, then
(a) $\alpha=1, \beta=-1$ (b) $\alpha=1, \beta=\pm 1$
(c) $\alpha=-1, \beta=\pm 1$ (d) $\alpha=\pm 1, \beta=1$

23. If $\vec{a}$ by any vector, then $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=$
(a) $(\vec{a})^2$ (b) $2(\vec{a})^2$
(c) $3(\vec{a})^2$ (d) 0

24. If the vectors $a\hat{i}+\hat{j}+\hat{k}, \hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ $(a\neq b, c\neq 1)$ are coplanar, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ is
(a) 1 (b) −1
(c) 2 (d) None of these

25. For any vector $\vec{A}$, the value of $\hat{i}\times(\vec{A}\times\hat{i})+\hat{j}\times(\vec{A}\times\hat{j})+\hat{k}\times(\vec{A}\times\hat{k})$ is equal to
(a) $\vec{0}$ (b) $2\vec{A}$
(c) $-2\vec{A}$ (d) None of these

26. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{p}, \vec{q}, \vec{r}$ are reciprocal system of vectors, then $\vec{a}\times\vec{p}+\vec{b}\times\vec{q}+\vec{c}\times\vec{r}$ equals
(a) $[\vec{a}\ \vec{b}\ \vec{c}]$ (b) $\vec{p}+\vec{q}+\vec{r}$
(c) $\vec{0}$ (d) $\vec{a}+\vec{b}+\vec{c}$

27. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit coplanar vectors, then the scalar triple product
$$[2\vec{a}-\vec{b}\ \ 2\vec{b}-\vec{c}\ \ 2\vec{c}-\vec{a}]=$$
(a) 0 (b) 1
(c) $-\sqrt{3}$ (d) $\sqrt{3}$

28. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors such that
$$\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}},$$
then the angle between $\vec{a}$ and $\vec{b}$ is
(a) $\pi/4$ (b) $\pi/2$
(c) π (d) $3\pi/4$

29. Let $\vec{a}(x)=(\sin x)\hat{i}+(\cos x)\hat{j}$ and $\vec{b}(x)=(\cos 2x)\hat{i}+(\sin 2x)\hat{j}$ be two variable vectors ($x\in R$), then $\vec{a}(x)$ and $\vec{b}(x)$ are
(a) collinear for unique value of x
(b) perpendicular for infinitely many values of x
(c) zero vectors for unique values of x
(d) none of these

30. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between P_1 and P_2
(a) 0 (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

31. Let $|\vec{a}|=1, |\vec{b}|=2, |\vec{c}|=3$ and $\vec{a}\perp(\vec{b}+\vec{c}), \vec{b}\perp(\vec{c}+\vec{a})$ and $\vec{c}\perp(\vec{a}+\vec{b})$. Then $|\vec{a}+\vec{b}+\vec{c}|$ is
(a) $\sqrt{6}$ (b) 6
(c) $\sqrt{14}$ (d) None of these

32. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 2 inclined at an angle 60° then the angle between $\vec{a}$ and $\vec{a}+\vec{b}$ is
(a) 30° (b) 60°
(c) 45° (d) None of these

33. $(\vec{r}.\hat{i})(\vec{r}\times\hat{i})+(\vec{r}.\hat{j})(\vec{r}\times\hat{j})+(\vec{r}.\hat{k})(\vec{r}\times\hat{k})$ is equal to
(a) $3\vec{r}$ (b) $\vec{r}$
(c) $\vec{0}$ (d) None of these

34. $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and $\vec{p}, \vec{q}, \vec{r}$ are defined as

$$\vec{p}=\frac{\vec{b}\times\vec{c}}{[\vec{b}\ \vec{c}\ \vec{a}]},\ \vec{q}=\frac{\vec{c}\times\vec{a}}{[\vec{c}\ \vec{a}\ \vec{b}]},\ \vec{r}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

then $(\vec{a}+\vec{b}).\vec{p}+(\vec{b}+\vec{c}).\vec{q}+(\vec{c}+\vec{a}).\vec{r}$ is equal to
(a) 0 (b) 1
(c) 2 (d) 3

35. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}.\vec{c}=|\vec{c}|, |\vec{c}-\vec{a}|=2\sqrt{2}$ and the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is 30° then $|(\vec{a}\times\vec{b})\times\vec{c}|$ is equal to
(a) 2/3 (b) 3/2
(c) 2 (d) 3

36. The number of vectors of unit length perpendicular to the vectors $\vec{a}=(1, 1, 0)$ and $\vec{b}=(0, 1,1)$ is
(a) 1 (b) 2
(c) 3 (d) infinite

37. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $\vec{a}.\vec{b}<0$ and $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$, then angle between vectors $\vec{a}$ & $\vec{b}$ is
(a) π (b) $7\pi/4$
(c) $\pi/4$ (d) $3\pi/4$

38. If $\vec{d}=\lambda(\vec{a}\times\vec{b})+\mu(\vec{b}\times\vec{c})+\nu(\vec{c}\times\vec{a})$ and $[\vec{a}\ \vec{b}\ \vec{c}]=1/8$, then $\lambda+\mu+\nu=$
(a) $\vec{d}\,(\vec{a}+\vec{b}+\vec{c})$ (b) $2\,\vec{d}.(\vec{a}+\vec{b}+\vec{c})$
(c) $4\,\vec{d}.(\vec{a}+\vec{b}+\vec{c})$ (d) $8\,\vec{d}\,(\vec{a}+\vec{b}+\vec{c})$

39. If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors, then $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$ if and only if
(a) $\vec{b}$ amd $\vec{c}$ are collinear
(b) $\vec{a}$ and $\vec{c}$ are collinear
(c) $\vec{a}$ and $\vec{b}$ are collinear
(d) None of these

40. The position vectors of the points A, B and C are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+5\hat{j}-\hat{k}$ and $2\hat{i}+3\hat{j}+5\hat{k}$ respectively. The greatest angle of the triangle ABC is
(a) 90° (b) 135°
(c) $\cos^{-1}\left(\frac{2}{3}\right)$ (d) $\cos^{-1}\left(\frac{5}{7}\right)$

41. Let $\vec{a}$ and $\vec{b}$ are two vectors making angle θ with each other, then unit vectors along bisector of $\vec{a}$ and $\vec{b}$ is
(a) $\pm\frac{\hat{a}+\hat{b}}{2}$ (b) $\pm\frac{\hat{a}+\hat{b}}{2\cos\theta}$
(c) $\pm\frac{\hat{a}+\hat{b}}{2\cos\theta/2}$ (d) $\pm\frac{(\hat{a}+\hat{b})}{|\hat{a}+\hat{b}|}$

42. The volume of the tetrahedron whose vertices are with position vectors $\hat{i}-6\hat{j}+10\hat{k}$, $-\hat{i}-3\hat{j}+7\hat{k}$, $5\hat{i}-\hat{j}+\lambda\hat{k}$ and $7\hat{i}-4\hat{j}+7\hat{k}$ is 11 cubic units if λ equals
(a) -3 (b) 3
(c) 7 (d) -1

43. If $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$ then
(a) $\vec{a}, \vec{b}, \vec{c}$ are orthogonal in pairs
(b) $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$
(c) $|\vec{a}|=|\vec{b}|=|\vec{c}|\neq 1$
(d) $|\vec{a}|\neq|\vec{b}|\neq|\vec{c}|$

44. A vector $\vec{a}$ has components $2p$ and 1 with respect to a rectangular cartesian system. This system is rotated through a certain angle about the origin in the clockwise sense. If with respect to new system, $\vec{a}$ has components $p+1$ and 1, then
(a) $p=0$
(b) $p=1$ or $p=-\frac{1}{3}$
(c) $p=-1$ or $p=\frac{1}{3}$
(d) $p=1$ or $p=-1$

45. If $\vec{a}$ and $\vec{b}$ are parallel then the value of $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{b}\times\vec{d})$ is equal to
(a) $\{(\vec{a}\times\vec{c}).\vec{b}\}\vec{d}$ (b) $\{(\vec{b}\times\vec{c}).\vec{a}\}\vec{d}$

(c) $\{(\vec{a}\times\vec{b})\cdot\vec{c}\}\vec{d}$ (d) None of these

46. If $\hat{i}\times(\vec{r}\times\hat{i})+\hat{j}\times(\vec{r}\times\hat{j})+\hat{k}\times(\vec{r}\times\hat{k})$
$=\vec{a}\times\vec{b}$ $(\vec{a}\neq 0,\ \vec{b}\neq 0)$, then

(a) $\vec{r}=\vec{a}\times\vec{b}$ (b) $\vec{r}=\dfrac{\vec{a}\times\vec{b}}{2}$

(c) $\vec{r}=\vec{0}$ (d) None of these

47. Let $\vec{a},\vec{b},\vec{c}$ be three vectors such that $\vec{a}\neq 0$ and $\vec{a}\times\vec{b}=2\vec{a}\times\vec{c}$, $|\vec{a}|=|\vec{c}|=1$, $|\vec{b}|=4$ and $|\vec{b}\times\vec{c}|=\sqrt{15}$, if $\vec{b}-2\vec{c}=\lambda\vec{a}$. Then λ equals

(a) 1 (b) -1

(c) 2 (d) -4

48. Let AD be the angle bisector of the angle A of ΔABC, then

$\vec{AD}=\alpha\vec{AB}+\beta\vec{AC}$, where

(a) $\alpha=\dfrac{|\vec{AB}|}{|\vec{AB}+\vec{AC}|}$, $\beta=\dfrac{|\vec{AC}|}{|\vec{AB}+\vec{AC}|}$

(b) $\alpha=\dfrac{|\vec{AB}|+|\vec{AC}|}{|\vec{AB}|}$, $\beta=\dfrac{|\vec{AB}|+|\vec{AC}|}{|\vec{AC}|}$

(c) $\alpha=\dfrac{|\vec{AC}|}{|\vec{AB}|+|\vec{AC}|}$, $\beta=\dfrac{|\vec{AB}|}{|\vec{AB}|+|\vec{AC}|}$

(d) $\alpha=\dfrac{|\vec{AB}|}{|\vec{AC}|}$, $\beta=\dfrac{|\vec{AC}|}{|\vec{AB}|}$

49. $p\hat{i}+3\hat{j}+4\hat{k}$ and $\sqrt{q}\,\hat{i}+4\hat{k}$ are two vectors, where $p, q>0$ are two scalars, then the length of the vectors is equal to

(a) All values of (p, q)

(b) Only finite number of values of (p, q)

(c) Infinite number of values of (p, q)

(d) No value of (p, q)

50. The vectors $\vec{p}=2\hat{i}+\log_3 x\hat{j}+a\hat{k}$ and $\vec{q}=-3\hat{i}+a\log_3 x\hat{j}+\log_3 x\hat{k}$ include an acute angle for

(a) $a=0$ (b) $a<0$

(c) $a>0$ (d) no real value of a

51. If $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=\ldots\{(\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}\}$

(a) -1 (b) 0

(c) 2 (d) None of these

52. If $\vec{a},\vec{b},\vec{c}$ are position vectors of three non-collinear points A, B, C respectively, the shortest distance from A to BC is

(a) $\vec{a}\cdot|\vec{b}-\vec{c}|\vec{a}$

(b) $\sqrt{|\vec{b}-\vec{a}|^2-\left[\dfrac{\vec{a}\cdot\vec{b}}{\vec{c}}\right]^2}$

(c) $|\vec{b}-\vec{a}|$

(d) None of these

53. If the non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other then the solution of the equation $\vec{r}\times\vec{a}=\vec{b}$ is

(a) $\vec{r}=x\vec{a}+\dfrac{1}{\vec{a}\cdot\vec{a}}(\vec{a}\times\vec{b})$

(b) $\vec{r}=x\vec{b}-\dfrac{1}{\vec{b}\cdot\vec{b}}(\vec{a}\times\vec{b})$

(c) $\vec{r}=x(\vec{a}\times\vec{b})$

(d) None of these

54. Let $\vec{OA}=\vec{a}$, $\vec{OB}=10\vec{a}+2\vec{b}$ and $\vec{OC}=\vec{b}$ where A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If $p=kq$, then $k=$

(a) 2 (b) 4

(c) 6 (d) None of these

55. Let $\vec{r},\vec{a},\vec{b}$ and $\vec{c}$ be four non-zero vectors such that $\vec{r}\cdot\vec{a}=0$, $|\vec{r}\times\vec{b}|=|\vec{r}||\vec{b}|$, $|\vec{r}\times\vec{c}|=|\vec{r}||\vec{c}|$ then $[\vec{a}\ \vec{b}\ \vec{c}]=$

(a) -1 (b) 0

(c) 1 (d) 2

56. Given a cube $ABCDA_1B_1C_1D_1$ with lower base $ABCD$, upper base $A_1B_1C_1D_1$ and the lateral edges AA_1, BB_1, CC_1 and DD_1; M and M_1 are the centres of the faces $ABCD$ and $A_1B_1C_1D_1$ respectively. O is a point on the line MM_1 such that

$$\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}=\vec{OM_1},$$

then $\vec{OM}=\lambda\vec{OM_1}$ if $\lambda=$

(a) 1/16 (b) 1/8

(c) 1/4 (d) 1/2

57. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero and non-coplanar vectors and $\vec{p}, \vec{q}$ and $\vec{r}$ be three vectors given by

$$\vec{p} = \vec{a} + \vec{b} - 2\vec{c},\ \vec{q} = 3\vec{a} - 2\vec{b} + \vec{c}$$

and $\vec{r} = \vec{a} - 4\vec{b} + 2\vec{c}$

If the volume of the parallelopiped determined by $\vec{a}, \vec{b}$ and $\vec{c}$ is V_1 and that of the parallelpiped determined by $\vec{p}, \vec{q}$ and $\vec{r}$ is V_2 then $V_2 : V_1 =$

(a) 3 : 1 (b) 7 : 1
(c) 11 : 1 (d) 15 : 1

58. A line L_1 passes through the point $3\hat{i}$ and parallel to the vector $-\hat{i}+\hat{j}+\hat{k}$ and another line L_2 passes through the point $\hat{i}+\hat{j}$ and parallel to the vector $\hat{i}+\hat{k}$, then point of intersection of the lines is

(a) $2\hat{i}+\hat{j}+\hat{k}$ (b) $2\hat{i}-2\hat{j}+\hat{k}$
(c) $\hat{i}+2\hat{j}-\hat{k}$ (d) None of these

59. The line joining the points $6\vec{a}-4\vec{b}-5\vec{c}$, $-4\vec{c}$ and the line joining the points $-\vec{a}-2\vec{b}-3\vec{c}$, $\vec{a}+2\vec{b}-5\vec{c}$ intersect at

(a) $2\vec{c}$ (b) $-4\vec{c}$
(c) $8\vec{c}$ (d) None of these

60. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be unit vectors. Suppose that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C} = 0$ and that the angle between $\vec{B}$ and $\vec{C}$ is $\pi/6$ then $\vec{A} = k(\vec{B}\times\vec{C})$ and $k =$

(a) ± 2 (b) ± 4
(c) ± 6 (d) 0

MULTIPLE CHOICE –II

Each question, in this part, has one or more than one correct answer (s). For each question write the letters a, b, c, d corresponding to the correct answer(s)

61. If a vector $\vec{r}$ satisfies the equation $\vec{r}\times(\hat{i}+2\hat{j}+\hat{k}) = \hat{i}-\hat{k}$, then $\vec{r}$ is equal to

(a) $\hat{i}+3\hat{j}+\hat{k}$ (b) $3\hat{i}+7\hat{j}+3\hat{k}$
(c) $\hat{j}+t(\hat{i}+2\hat{j}+\hat{k})$ where t is any scalar
(d) $\hat{i}+(t+3)\hat{j}+\hat{k}$ where t is any scalar

62. If $\vec{DA} = \vec{a}$, $\vec{AB} = \vec{b}$ and $\vec{CB} = k\vec{a}$, where $k > 0$ and X, Y are the mid points of DB and AC respectively such that $|\vec{a}| = 17$ and $|\vec{XY}| = 4$, then k is equal to

(a) 8/17 (b) 9/17
(c) 25/17 (d) 4/17

63. $\vec{a}$ and $\vec{c}$ are unit vectors and $|\vec{b}| = 4$ with $\vec{a}\times\vec{b} = 2\vec{a}\times\vec{c}$. The angle between $\vec{a}$ and $\vec{c}$ is $\cos^{-1}(1/4)$. Then $\vec{b}-2\vec{c} = \lambda\vec{a}$, if λ is

(a) 3 (b) 1/4
(c) – 4 (d) – 1/4

64. If I be the incentre of the triangle ABC and a, b, c be the lengths of the sides then the force $a\,\vec{IA} + b\,\vec{IB} + c\,\vec{IC} =$

(a) –1 (b) 2
(c) 0 (d) None of these

65. If $3\vec{a}-5\vec{b}$ and $2\vec{a}+\vec{b}$ are perpendicular to each other and $\vec{a}+4\vec{b}$, $-\vec{a}+\vec{b}$ are also mutually perpendicular then the cosine of the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{17}{5\sqrt{43}}$ (b) $\frac{19}{5\sqrt{43}}$
(c) $\frac{21}{5\sqrt{43}}$ (d) None of these

66. A vector $\vec{a} = (x, y, z)$ makes an obtuse angle with y-axis, equal angles with $\vec{b} = (y, -2z, 3x)$ and $\vec{c} = (2z, 3x, -y)$ and $\vec{a}$ is perpendicular to $\vec{d} = (1, -1, 2)$ if $|\vec{a}| = 2\sqrt{3}$, then vector $\vec{a}$ is

(a) (1, 2, 3) (b) (2, –2, –2)
(c) (–1, 2, 4) (d) None of these

67. The vector $\vec{c}$ directed along the bisectors of the angle between the vecotrs $\vec{a} = 7\hat{i}-4\hat{j}-4\hat{k}$ and $\vec{b} = -2\hat{i}-\hat{j}+2\hat{k}$ if $|\vec{c}| = 3\sqrt{6}$ is given by

(a) $\hat{i}-7\hat{j}+2\hat{k}$
(b) $\hat{i}+7\hat{j}-2\hat{k}$
(a) $-\hat{i}+7\hat{j}-2\hat{k}$
(d) $\hat{i}-7\hat{j}-2\hat{k}$

68. Let $\vec{a}$ and $\vec{b}$ be non collinear vectors of which $\vec{a}$ is a unit vector. The angles of the triangle whose two sides are represented by $\sqrt{3}\,(\vec{a}\times\vec{b})$ and $\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{a}$ are

(a) $\pi/2, \pi/3, \pi/6$ (b) $\pi/2, \pi/4, \pi/4$

(c) $\pi/3, \pi/3, \pi/3$ (d) data insufficient

69. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\,\vec{b}$ and $|\vec{v}|=\vec{a}\times\vec{b}$, then $|\vec{v}|$ is

(a) $|\vec{u}|$ (b) $|\vec{u}|+|\vec{u}\cdot\vec{a}|$

(c) $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ (b) $|\vec{u}|+\vec{u}\cdot(\vec{a}+\vec{b})$

70. If A, B, C are three points with position vectors $\hat{i}+\hat{j}$, $\hat{i}-\hat{j}$ and $p\,\hat{i}+q\,\hat{i}+r\,\hat{k}$ respectively, then the points are collinear if

(a) $p=q=r=1$ (b) $p=q=r=0$

(c) $p=q, r=0$ (d) $p=1, q=2, r=0$

71. In a parallelogram $ABCD$, $|\vec{AB}|=a$, $|\vec{AD}|=b$ and $|\vec{AC}|=c$. Then $\vec{DB}\cdot\vec{AB}$ has the value

(a) $\dfrac{3a^2+b^2-c^2}{2}$ (b) $\dfrac{a^2+3b^2-c^2}{2}$

(c) $\dfrac{a^2-b^2+3c^2}{2}$ (d) $\dfrac{a^2+3b^2+c^2}{2}$

72. If $|\vec{a}|=4$, $|\vec{b}|=2$ and the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$ then $(\vec{a}\times\vec{b})^2$ is

(a) 48 (b) $(\vec{a})^2$

(c) 16 (d) 32

73. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar non-zero vectors and $\vec{r}$ is any vector in space then $[\vec{b}\ \vec{c}\ \vec{r}]\,\vec{a}+[\vec{c}\ \vec{a}\ \vec{r}]\,\vec{b}+[\vec{a}\ \vec{b}\ \vec{r}]\,\vec{c}$ is equal to

(a) $3\,[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{r}$ (b) $[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{r}$

(c) $[\vec{b}\ \vec{c}\ \vec{a}]\,\vec{r}$ (d) $[\vec{c}\ \vec{a}\ \vec{b}]\,\vec{r}$

74. If the unit vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle 2θ such that $|\vec{a}-\vec{b}|<1$ and $0<\theta<\pi$, then θ lies in the interval

(a) $[0, \pi/6)$ (b) $(5\pi/6, \pi]$

(c) $[\pi/6, \pi/2]$ (d) $[\pi/2, 5\pi/6]$

75. The vectors $2\,\hat{i}-\lambda\,\hat{j}+3\lambda\,\hat{k}$ and $(1+\lambda)\,\hat{i}-2\lambda\,\hat{j}+\hat{k}$ include an acute angle for

(a) all values of m (b) $\lambda<-2$

(c) $\lambda>-1/2$ (d) $\lambda\in[-2,-1/2]$

76. The vectors $\vec{a}=x\,\hat{i}-2\,\hat{j}+5\,\hat{k}$ and $\vec{b}=\hat{i}+y\,\hat{j}-z\,\hat{k}$ are collinear if

(a) $x=1, y=-2, z=-5$

(b) $x=1/2, y=-4, z=-10$

(c) $x=-1/2, y=4, z=10$

(d) $x=-1, y=2, z=5$

77. Let $\vec{a}=2\,\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\,\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\,\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is of magnitude $\sqrt{(2/3)}$ is

(a) $2\,\hat{i}+3\,\hat{j}-3\,\hat{k}$ (b) $2\,\hat{i}+3\,\hat{j}+3\,\hat{k}$

(c) $-2\,\hat{i}-\hat{j}+5\,\hat{k}$ (d) $2\,\hat{i}+\hat{j}+5\,\hat{k}$

78. The vectors

$(x, x+1, x+2)$, $(x+3, x+4, x+5)$

and $(x+6, x+7, x+8)$ are coplanar for

(a) all values of x (b) $x<0$

(c) $x>0$ (d) None of these

79 If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that

$$\vec{r}_1=\vec{a}-\vec{b}+\vec{c},\ \vec{r}_2=\vec{b}+\vec{c}-\vec{a},$$
$$\vec{r}_3=\vec{c}+\vec{a}+\vec{b},\ \vec{r}=2\,\vec{a}-3\,\vec{b}+4\,\vec{c}\ \text{if}$$
$$\vec{r}=\lambda_1\,\vec{r}_1+\lambda_2\,\vec{r}_2+\lambda_3\,\vec{r}_3,\ \text{then}$$

(a) $\lambda_1=7/2$ (b) $\lambda_1+\lambda_2=3$

(c) $\lambda_1+\lambda_2+\lambda_3=4$ (d) $\lambda_2+\lambda_3=2$

80. A parallelogram is constructed on the vectors $\vec{a}=3\,\vec{\alpha}-\vec{\beta}$, $\vec{b}=\vec{\alpha}+3\,\vec{\beta}$ if $|\vec{\alpha}|=|\vec{\beta}|=2$ and angle between $\vec{\alpha}$ and $\vec{\beta}$ is $\pi/3$ then the length of a diagonal of the parallelogram is

(a) $4\sqrt{5}$ (b) $4\sqrt{3}$

(c) $4\sqrt{7}$ (d) None of these

81. The position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ of four points A, B, C and D on a plane are such that $(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c})=(\vec{b}-\vec{d})\cdot(\vec{c}-\vec{a})=0$, then the point D is

(a) Centroid of $\Delta\,ABC$

(b) Orthocentre of $\Delta\,ABC$

(c) Circumcentre of $\Delta\,ABC$

(d) None of these

82. The vector $\vec{a}+\vec{b}$ bisects the angle between the vectors $\hat{a}$ and $\hat{b}$ if
(a) $|\vec{a}|=|\vec{b}|$
(b) angle between $\vec{a}$ and $\vec{b}$ is zero
(c) $|\vec{a}|+|\vec{b}|=0$
(d) None of these

83. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that $\vec{b}$ is not perpendicular to both $\vec{a}$ and $\vec{c}$ and $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ then
(a) $\vec{a}$ and $\vec{c}$ are always non-collinear
(b) $\vec{a}$ and $\vec{c}$ are always collinear
(c) $\vec{a}$ and $\vec{c}$ are always perpendicular
(d) $\vec{a}, \vec{b}$ and $\vec{c}$ are always non-coplanar

84. Image of the point P with position vector $7\hat{i}-\hat{j}+2\hat{k}$ in the line whose vector equation is $\vec{r}=9\hat{i}+5\hat{j}+5\hat{k}+\lambda(\hat{i}+3\hat{j}+5\hat{k})$ has the position vector
(a) $-9\hat{i}+5\hat{j}+2\hat{k}$
(b) $9\hat{i}+5\hat{j}-2\hat{k}$
(c) $9\hat{i}-5\hat{j}-2\hat{k}$
(d) $9\hat{i}+5\hat{j}+2\hat{k}$

85. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, then
(a) $(\vec{a}-\vec{d})=\lambda(\vec{b}-\vec{c})$
(b) $\vec{a}+\vec{d}=\lambda(\vec{b}+\vec{c})$
(c) $(\vec{a}-\vec{b})=\lambda(\vec{c}+\vec{d})$
(d) None of these

86. If $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$, then
(a) $\vec{b}\times(\vec{c}\times\vec{a})=0$
(b) $(\vec{c}\times\vec{a})\times\vec{b}=0$
(c) $\vec{c}\times(\vec{a}\times\vec{b})=0$
(d) None of these

87. If the vectors $\vec{b}=(\tan\alpha, -1, 2\sqrt{\sin\alpha/2})$ and $\vec{c}=\left(\tan\alpha, \tan\alpha, -\frac{3}{\sqrt{\sin\alpha/2}}\right)$ are orthogonal and a vector $\vec{a}=(1, 3, \sin 2\alpha)$ makes an obtuse angle with the z-axis, then the value of α is
(a) $\alpha=(4n+1)\pi-\tan^{-1}2$
(b) $\alpha=(4n+2)\pi-\tan^{-1}2$
(c) $\alpha=(4n+1)\pi+\tan^{-1}2$
(d) $\alpha=(4n+2)\pi+\tan^{-1}2$

88. The resolved part of the vector $\vec{a}$ along the vector $\vec{b}$ is $\vec{\lambda}$ and that perpendicular to $\vec{b}$ is $\vec{\mu}$. Then
(a) $\vec{\lambda}=\frac{(\vec{a}\cdot\vec{b})\vec{a}}{(\vec{a})^2}$
(b) $\vec{\lambda}=\frac{(\vec{a}\cdot\vec{b})\vec{b}}{(\vec{b})^2}$
(c) $\vec{\mu}=\frac{(\vec{b}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}}{(\vec{b})^2}$
(d) $\vec{\mu}=\frac{\vec{b}\times(\vec{a}\times\vec{b})}{(\vec{b})^2}$

89. Let the unit vectors $\vec{a}$ and $\vec{b}$ be perpendicular and the unit vector $\vec{c}$ be inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}+\vec{b})$ then
(a) $\alpha=\beta$
(b) $\gamma^2=1-2\alpha^2$
(c) $\gamma^2=-\cos 2\theta$
(d) $\beta^2=\frac{1+\cos 2\theta}{2}$

90. Consider a tetrahedran with faces F_1, F_2, F_3, F_4. Let $\vec{V_1}, \vec{V_2}, \vec{V_3}, \vec{V_4}$ be the vectors whose magnitudes are respectively equal to areas of F_1, F_2, F_3, F_4 and whose directions are perpendicular to their faces in outward direction. Then $|\vec{V_1}+\vec{V_2}+\vec{V_3}+\vec{V_4}|$ equals
(a) 1 (b) 4
(c) 0 (d) None of these

Practice Test

MM : 20 Time 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[10 × 2 = 20]

1. Let the unit vectors $\vec{a}$ and $\vec{b}$ be perpendicular to each other and the unit vector $\vec{c}$ be inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c} = x\vec{a} + y\vec{b} + z(\vec{a} \times \vec{b})$, then

(a) $x = \cos\theta,\ y = \sin\theta,\ z = \cos 2\theta$

(b) $x = \sin\theta,\ y = \cos\theta,\ z = -\cos 2\theta$

(c) $x = y = \cos\theta,\ z^2 = \cos 2\theta$

(d) $x = y = \cos\theta,\ z^2 = -\cos 2\theta$

2. $\Delta = \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c} \end{vmatrix}$, then

(a) $\Delta = 0$

(b) $\Delta = 1$

(c) Δ = any non-zero value

(d) None of these

3. The image of a point $2\hat{i} + 2\hat{j} - \hat{k}$ in the line passing through the points $\hat{i} - \hat{j} + 2\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$ is

(a) $3\hat{i} + 11\hat{j} + 7\hat{k}$ (b) $\dfrac{-\hat{i} - 11\hat{j} + 7\hat{k}}{3}$

(c) $-2\hat{i} - 2\hat{j} + \hat{k}$ (d) None of these

4. Let O be the circumcentre, G be the centroid and O' be the orthocentre of a ΔABC. Three vectors are taken through O and are represented by $\vec{a} = \vec{OA}, \vec{b} = \vec{OB}$ and $\vec{c} = \vec{OC}$ then $\vec{a} + \vec{b} + \vec{c}$ is

(a) $\vec{OG}$ (b) $2\vec{OG}$

(c) $\vec{OO'}$ (d) None of then

5. If the vectors $\vec{c}, \vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{b} = \hat{j}$ are such that $\vec{a}, \vec{c}$ and $\vec{b}$ form a right handed system, then $\vec{c}$ is

(a) $z\hat{i} - x\hat{k}$ (b) $\vec{o}$

(c) $y\hat{j}$ (d) $-z\hat{i} + x\hat{k}$

6. The equation of the plane containing the line $\vec{r} = \vec{a} + k\vec{b}$ and perpendicular to the plane $\vec{r}\cdot\vec{n} = q$ is

(a) $(\vec{r} - \vec{b})\cdot(\vec{n} \times \vec{a}) = 0$

(b) $(\vec{r} - \vec{a})\cdot\{\vec{n} \times (\vec{a} \times \vec{b})\} = 0$

(c) $(\vec{r} - \vec{a})\cdot(\vec{n} \times \vec{b}) = 0$

(d) $(\vec{r} - \vec{b})\cdot\{\vec{n} \times (\vec{a} \times \vec{b})\} = 0$

7. Find the value of λ so that the points P, Q, R, S on the side OA, OB, OC and AB of a regular tetrahedron are coplanar. You are given that $\dfrac{\vec{OP}}{\vec{OA}} = \dfrac{1}{3}$;

$\dfrac{\vec{OQ}}{\vec{OB}} = \dfrac{1}{2}, \dfrac{\vec{OR}}{\vec{OC}} = \dfrac{1}{3}$ and $\dfrac{\vec{OS}}{\vec{AB}} = \lambda$

(a) $\lambda = 1/2$ (b) $\lambda = -1$

(c) $\lambda = 0$ (d) for no value of λ

8. $\hat{x}$ and $\hat{y}$ are two mutually perpendicular unit vectors, if the vectors $a\hat{x} + a\hat{y} + c(\hat{x} \times \hat{y})$, $\hat{x} + (\hat{x} + \hat{y})$ and $c\hat{x} + c\hat{y} + b(\hat{x} \times \hat{y})$, lie in a plane then c is

(a) A.M. of a and b

(b) G.M. of a and b

(c) H.M. of a and b

(d) equal to zero

9. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j}$, then the vectors $(\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}$, $(\vec{b}\cdot\hat{i})\hat{i} + (\vec{b}\cdot\hat{j})\hat{j} + (\vec{b}\cdot\hat{k})\hat{k}$, and $\hat{i} + \hat{j} - 2\hat{k}$

(a) are mutually perpendicular

(b) are coplanar

(c) Form a parallelopiped of volume 6 units

(d) Form a parallelopiped of volume 3 units

10. If unit vectors $\hat{i}$ and $\hat{j}$ are at right angles to each other and $\vec{p} = 3\hat{i} + 4\hat{j}$, $\vec{q} = 5\hat{i}$, $4\vec{r} = \vec{p} + \vec{q}$, then $2\vec{s} = \vec{p} - \vec{q}$

(a) $|\vec{r} + k\vec{s}| = |\vec{r} - k\vec{s}|$ for all real k

(b) $\vec{r}$ is perpendicular to $\vec{s}$

(c) $\vec{r} + \vec{s}$ is perpendicular to $\vec{r} - \vec{s}$

(d) $|\vec{r}| = |\vec{s}| = |\vec{p}| = |\vec{q}|$

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answer

Multiple Choice–I

1. (d)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (d)
7. (a)	**8.** (c)	**9.** (b)	**10.** (a)	**11.** (c)	**12.** (c)
13. (d)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (d)	**18.** (a)
19. (c)	**20.** (d)	**21.** (d)	**22.** (d)	**23.** (b)	**24.** (a)
25. (b)	**26.** (c)	**27.** (a)	**28.** (d)	**29.** (b)	**30.** (c)
31. (a)	**32.** (a)	**33.** (c)	**34.** (d)	**35.** (b)	**36.** (b)
37. (d)	**38.** (d)	**39.** (b)	**40.** (a)	**41.** (c)	**42.** (c)
43. (b)	**44.** (b)	**45.** (d)	**46.** (b)	**47.** (d)	**48.** (c)
49. (c)	**50.** (d)	**51.** (c)	**52.** (d)	**53.** (a)	**54.** (c)
55. (b)	**56.** (c)	**57.** (d)	**58.** (a)	**59.** (b)	**60.** (a)

Multiple Choice–II

61. (a), (b), (c)	**62.** (b), (c)	**63.** (a), (c)	**64.** (c)	**65.** (b)	**66.** (b)
67. (a), (c)	**68.** (a)	**69.** (a), (c)	**70.** (c), (d)	**71.** (a)	**72.** (b), (c)
73. (b), (c), (d)	**74.** (a), (b)	**75.** (b), (c)	**76.** (a), (b), (c), (d)		**77.** (a), (c)
78. (a), (b), (c)	**79.** (a), (c)	**80.** (b), (c)	**81.** (b)	**82.** (a), (b)	**83.** (b)
84. (b)	**85.** (a), (b)	**86.** (a), (b)	**87.** (a), (b)	**88.** (b), (c), (d)	
89. (a), (b), (c), (d)	**90.** (c).				

Practice Test

1. (d)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (c)
7. (b)	**8.** (c)	**9.** (a), (c)	**10.** (a), (b), (c)		

33

CO-ORDINATE GEOMETRY–3D

1. The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is space in given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Corollary 1. Distance of $(x_1, y_1, z_1,)$ from origin $= \sqrt{(x_1^2 + y_1^2 + z_1^2)}$

2. Section formula : If $R(x, y, z)$ divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2,)$ in the ratio $m_1 : m_2$ ($m_1, m_2 > 0$) then

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}; y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}; z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

(divides internally)

and
$$x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}; y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}; z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$

(divides externally)

Corollary 1. If $R(x, y, z)$ divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $\lambda : 1$ then

$$x = \frac{\lambda x_2 \pm x_1}{\lambda + 1}, y = \frac{\lambda y_2 \pm y_1}{\lambda \pm 1}; z = \frac{\lambda z_2 \pm z_1}{\lambda \pm 1}$$

positive sign is taken for internal division and negative sign is taken for external division.

Corollary 2. The mid point of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

3. Centroid of a Triangle : The centroid of a triangle ABC whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3\ z_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

4. Centroid of a Tetrahedron : The centroid of a tetrahedron $ABCD$ whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

5. Direction Cosines (D.C.'s) : If a line makes an angles α, β, γ with positive directions of x, y and z axes then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines (or d.c.'s) of the line Generally direction cosines are represented by l, m, n.

Then angles α, β, γ are called the **direction angles** of the line AB & the direction cosines of BA are $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$ and $\cos(\pi - \gamma)$ *i.e.*, $-\cos\alpha, -\cos\beta, -\cos\gamma$.

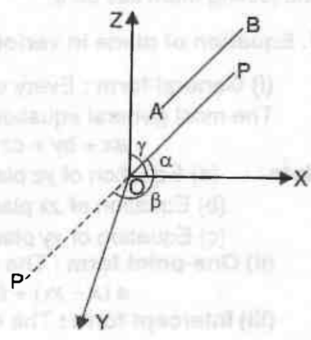


Fig.

Corollary 1. The direction cosines of the x-axis are $\cos 0, \cos \pi/2, \cos \pi/2$ *i.e.*, 1, 0, 0.

Similarly the d.c.'s of y and z axis are (0, 1, 0) and (0, 0, 1) respectively.

Corollary 2. If l, m, n be the d.c.'s of a line OP and $OP = r$, then the co-ordinates of the point P are (lr, mr, nr).

Corollary 3. $l^2 + m^2 + n^2 = 1$

or
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Corollary 4. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

6. Direction ratios (d.r.'s) : Direction ratios of a line are numbers which are proportional to the d.c.'s of a line.

Direction ratios of a line PQ, (where P and Q are (x_1, y_1, z_1) & (x_2, y_2, z_2) respectively, are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

7. Relation between the d.c.'s and d.r.'s : If a, b, c are the d.r.'s and l, m, n are the d.c.'s, then

$$l = \pm \frac{a}{\sqrt{(a^2+b^2+c^2)}},\ m = \pm \frac{b}{\sqrt{(a^2+b^2+c^2)}},\ n = \pm \frac{c}{\sqrt{(a^2+b^2+c^2)}}$$

Note : If a, b, c are the d.r.'s of AB then d.c.'s of AB are given by the + *ve* sign and those of the line BA by − *ve* sign.

8. The angle between two lines : If (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction cosines of any two lines and θ be the angle between them, then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Corollary 1. If lines are perpendicular then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

Corollary 2. If lines are parallel then

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

Corollary 3. If the d.r.'s of the two lines are a_1, b_1, c_1 and a_2, b_2, c_2 then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

&

$$\sin\theta = \pm \frac{\sqrt{\Sigma (b_1c_2 - b_2c_1)^2}}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

So that the conditions for perpendicular and parallelism of two lines are respectively.

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \text{ and } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Corollary 4. If l_1, m_1, n_1 & l_2, m_2, n_2 are the d.c.'s of two lines, the d.r.'s of the line which are perpendicular to both of them are : $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.

9. Projection of a line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m, n.

$$= |(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n|$$

Corollary 1. If P is a point (x_1, y_1, z_1) then the projection of OP on a line whose direction cosines are l_1, m_1, n_1 is $|l_1x_1 + m_1y_1 + n_1z_1|$ where O is origin.

Corollary 2. The projections of PQ when P is (x_1, y_1, z_1) and Q is (x_2, y_2, z_2) on the co-ordinates axes are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Corollary 3. If Projections of PQ on AB is zero then PQ is perpendicular to AB.

THE PLANE

A plane is defined as the surface such that if any two points on it are taken as then every point on the line joining them lies on it.

1. Equation of plane in various forms

(i) General form : Every equation of first degree in x, y, z represent a plane.

The most general equation of the first degree in x, y, z is

$$ax + by + cz + d = 0, \text{ where } a, b, c \text{ are not all zero.}$$

Note : (a) Equation of yz plane is $x = 0$

(b) Equation of zx plane is $y = 0$

(c) Equation of xy plane is $z = 0$

(ii) One-point form : The equation of plane through (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

(iii) Intercept form : The equation of plane in terms of intercepts of a, b, c from the axes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(iv) Normal form : The equation of plane on which the perpendicular from origin of length p and the direction cosines of perpendicular are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ with the positive directions of x, y & z axes respectively is given by

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p$$

(v) Equation of plane passing through three given points : Equation of plane passing through $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{vmatrix} = 0$$

(vi) Equation of a plane passing through a point and parallel to two lines : The equation of the plane passing through a point $P(x_1, y_1, z_1)$ and parallel to two lines whose d.c.'s and l_1, m_1, n_1 and l_2, m_2, n_2 is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(vii) Equation of a plane passing through two points and parallel to a line : The equation of the plane passes through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and is parallel to a line whose d.c.'s are l, m, n is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$$

2. Angle Between two Planes :

If θ be the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ then

$$\theta = \cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}\right)$$

Corollary 1. If planes are perpendicular then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Corollary 2. If planes are parallel then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

3. Angle Between a Plane and a Line :

If α be the angle between the normal to the plane and a line then $90° - \alpha$ is the angle between the plane and the line.

4. Length of Perpendicular from a Point to a Plane

The length of perpendicular from (x_1, y_1, z_1) on $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$$

5. Positions of Points (x_1, y_1, z_1) and (x_2, y_2, z_2) relative to a Plane :

If the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are on the same side or opposite side of the plane $ax + by + cz + d = 0$ then

$$\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0 \text{ or } \frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} < 0$$

6. Distance between the Parallel Planes :

Let two parallel planes be $ax + by + cz + d = 0$ and $ax + by + cz + d_1 = 0$

First Method : The distance between parallel planes is

$$\frac{|d - d_1|}{\sqrt{(a^2 + b^2 + c^2)}}$$

Second Method : Find the co-ordinates of any point on one of the given planes, preferably putting $x = 0$, $y = 0$ or $y = 0$, $z = 0$ or $z = 0$, $x = 0$. Then the perpendicular distance of this point from the other plane is the required distance between the planes.

7. Family of Planes

Any plane passing through the line of intersection of the planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ can be represented by the equation

$$(ax + by + cz + d) + \lambda (a_1x + b_1y + c_1z + d_1) = 0$$

8. Equations of Bisectors of the Angles between two Planes :

Equations of the bisectors of the planes

$$P_1 : ax + by + cz + d = 0 \ \& \ P_2 = a_1x + b_1y + c_1z + d_1 = 0$$

(where $d > 0$ & $d_1 > 0$) are

$$\frac{(ax + by + cz + d)}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

Conditions	Acute angle Bisector	Obtuse angle Bisector
$aa_1 + bb_1 + cc_1 > 0$	−	+
$aa_1 + bb_1 + cc_1 < 0$	+	−

9. The Image of a Point with respect to Plane Mirror :

The image of $A(x_1, y_1, z_1)$ with respect to the plane mirror $ax + by + cz + d = 0$ be $B(x_2, y_2, z_2)$ is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{(a^2 + b^2 + c^2)}$$

10. The feet of perpendicular from a point on a plane

The feet of perpendicular from a point $A(x_1, y_1, z_1)$ on the plane $ax + by + cz + d = 0$ be $B(x_2, y_2, z_2)$ is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - b_1}{c} = \frac{-(ax_1 + by_1 + cz_1 + d)}{(a^2 + b^2 + c^2)}$$

11. Reflection of a plane on another plane :

The reflection of the plane $ax + by + cz + d = 0$ on the place $a_1x + b_1y + c_1z + d_1 = 0$ is

$$2(aa_1 + bb_1 + cc_1)(a_1x + b_1y + c_1z + d_1) = (a_1^2 + b_1^2 + c_1^2)(ax + by + cz + b)$$

12. Area of a Triangle

If A_{yz}, A_{zx}, A_{xy} be the projections of an area A on the co-ordinate planes yz, zx and xy respectively, then

$$A = \sqrt{(A_{yz}^2 + A_{zx}^2 + A_{xy}^2)}$$

If vertices of a triangle are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) then

$$A_{yz} = \frac{1}{2} \left| \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \right|$$

$$A_{zx} = \frac{1}{2} \left| \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} \right|$$

& $$A_{xy} = \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

Corollary : Area of triangle = $\frac{1}{2} bc \sin A$.

PAIR OF PLANES

1. Homogeneous Equation of Second degree :

An equation of the form

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is called a homogeneous equation of second degree. It represent two planes passing through origin. Condition that it represents a plane is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

i.e., $$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

2. Angle between two Planes :

If θ is the acute angle between two planes whose joint equation is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, then

$$\theta = \tan^{-1}\left\{ \frac{2\sqrt{f^2 + g^2 + h^2 - bc - (ca - ab)}}{a + b + c} \right\}$$

Corollary : If planes are perpendicular then

$$a + b + c = 0$$

THE STRAIGHT LINE

Straight line is the locus of the intersection of any two planes.

1. Equation of a straight line (General form) :

Let $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ be the equations of any two planes, taken together then

$ax + by + cz + d = 0 = a_1x + b_1y + c_1z + d_1$ is the equation of straight line.

Corollary : The *x*-axis has equations $y = 0 = z$, the *y*-axis $z = 0 = x$ and the *z*-axis $x = 0 = y$.

2. Equation of a line Passing through a Point and Parallel to a Specified Direction :

The equation of line passing through (x_1, y_1, z_1) and parallel to a line whose d.r.'s an a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r \text{ (say)}$$

& the co-ordinate of any point on the line an $(x_1 + ar, y_1 + br, z_1 + cr)$ when r is directed distance.

3. Equation of line Passing through two Points :

The equations of line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

4. Symmetric Form :

The equation of line passing through (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

5. To convert from General Equation of a line to Symmetrical Form :

(i) Point : Put $x=0$ (or $y=0$ or $z=0$) in the given equations and solve for y and z. The values of x, y and z are the co-ordinates of a point lying on the line.

(ii) Direction cosines : Since line is perpendicular to the normals to the given planes then find direction cosines. Then write down the equation of line with the help of a point & direction cosines.

6. Angle between a Line and a Plane :

If angle between the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $a_1x+b_1y+c_1z+d=0$ is θ then $90^\circ-\theta$ is the angle between normal and the line

i.e.,
$$\cos(90^\circ-\theta)=\frac{(aa_1+bb_1+cc_1)}{\sqrt{(a^2+b^2+c^2)}\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

or
$$\sin\theta=\frac{(aa_1+bb_1+cc_1)}{\sqrt{(a^2+b^2+c^2)}\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

Corollary : If line is parallel to the plane then $aa_1+bb_1+cc_1=0$

7. General Equation of the Plane Containing the Line :

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is
$$a(x-x_1)+b(y-y_1)+c(z-z_1)$$

where $al+bm+cn=0$

8. Coplanar lines :

(i) Equations of both lines in symmetrical form :

If the two lines are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}\ \&\ \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

Coplanar then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

& the equation of plane containing the line is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

(ii) If one line in symmetrical form & other in general form :

Let lines are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\ \text{and}$$

$$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$$

The condition for coplanarity is

$$\frac{a_1x_1+b_1y_1+c_1z_1+d_1}{a_2x_1+b_2y_1+c_2z_1+d_2}=\frac{a_1l+b_1m+c_1n}{a_2l+b_2m+c_2n}$$

(iii) If both line in General form :

Let lines are $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$

and $a_3x+b_3y+c_3z+d_3=0=a_4x+b_4y+c_4z+d_4$

The condition that this pair of lines is coplanar is

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

9. Skew Lines :

Two straight lines in space are called skew lines, when they are not coplanar. Thus skew lines are neither parallel, nor intersect at any point.

10. Shortest Distance (S.D.)

Let PQ and RS are two skew lines and a line which is perpendicular to both PQ and RS. Then the length of the line is called the shortest distance between PQ and RS.

Let equations of the given lines are

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

Let S.D. lie along the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

$\therefore$ $$\text{S.D.} = | l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1) |$$

and **Equation of shortest distance is**

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \ \& \ \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

11. Volume of Tetrahedron

If vertices of tetrahedron are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

MULTIPLE CHOICE

Each question in this part has four choices out of which just one is correct. Indicate your choice of correct answer for each question by writing one of the letters a, b, c, d whichever is appropriate.

1. The four lines drawn from the vertices of any tetrahedron to the centroid of the opposite faces meet in a point whose distance from each vertex is k times the distance from each vertex to the opposite face, where k is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $\frac{3}{4}$ (d) $\frac{5}{4}$

2. Which of the statement is true ? The coordinate planes divide the line joining the points (4, 7, –2) and (–5, 8, 3)

(a) all externally

(b) two externally and one internally

(c) two internally and one externally

(d) none of these

3. The pair of lines whose direction cosines are given by the equations $3l + m + 5n = 0$, $6mn - 2nl + 5lm = 0$ are

(a) parallel (b) perpendicular

(c) inclined at $\cos^{-1}\left(\frac{1}{6}\right)$

(d) none of these

4. The distance of the point $A(-2, 3, 1)$ from the line PQ through $P(-3, 5, 2)$ which make equal angles with the axes is

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{\frac{14}{3}}$

(c) $\frac{16}{\sqrt{3}}$ (d) $\frac{5}{\sqrt{3}}$

5. The equation of the plane through the point (2, 5, –3) perpendicular to the planes $x + 2y + 2z = 1$ and $x - 2y + 3z = 4$ is
(a) $3x - 4y + 2z - 20 = 0$
(b) $7x - y + 5z = 30$
(c) $x - 2y + z = 11$
(d) $10x - y - 4z = 27$

6. The equation of the plane through the points (0, –4, –6) and (–2, 9, 3) and perpendicular to the plane $x - 4y - 2z = 8$ is
(a) $3x + 3y - 2z = 0$
(b) $x - 2y + z = 2$
(c) $2x + y - z = 2$
(d) $5x - 3y + 2z = 0$

7. The equation of the plane passing through the points (3, 2, –1), (3, 4, 2) and (7, 0, 6) is $5x + 3y - 2z = \lambda$ where λ is
(a) 23 (b) 21
(c) 19 (d) 27

8. A variable plane which remains at a constant distance p from the origin cuts the coordinate axes in A, B, C. The locus of the centroid of the tetrahedron $OABC$ is $y^2z^2 + z^2x^2 + x^2y^2 = kx^2y^2z^2$ where k is equal to
(a) $9p^2$ (b) $\frac{9}{p^2}$
(c) $\frac{7}{p^2}$ (d) $\frac{16}{p^2}$

9. The line joining the points (1, 1, 2) and (3, –2, 1) meets the plane $3x + 2y + z = 6$ at the point
(a) (1, 1, 2) (b) (3, –2, 1)
(c) (2, –3, 1) (d) (3, 2, 1)

10. The point on the line $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+5}{-2}$ at a distance of 6 from the point (2, –3, –5) is
(a) (3, –5, –3) (b) (4, –7, –9)
(c) (0, 2, –1) (d) (–3, 5, 3)

11. The plane passing through the point (5, 1, 2) perpendicular to the line $2(x - 2) = y - 4 = z - 5$ will meet the line in the point
(a) (1, 2, 3) (b) (2, 3, 1)
(c) (1, 3, 2) (d) (3, 2, 1)

12. The point equidistant from the four points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$ is
(a) $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ (b) (a, b, c)
(c) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (d) None of these

13. P, Q, R, S are four coplanar points on the sides AB, BC, CD, DA of a skew quadrilateral. The product $\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA}$ equals
(a) –2 (b) –1
(c) 2 (d) 1

14. The angle between any two diagonals of a cube is
(a) $\cos\theta = \frac{\sqrt{3}}{2}$ (b) $\cos 0 = \frac{1}{\sqrt{2}}$
(c) $\cos\theta = \frac{1}{3}$ (d) $\cos\theta = \frac{1}{\sqrt{6}}$

15. The acute angle between two lines whose direction cosines are given by the relation between $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is
(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) None of these

16. The lines $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{-1}$, $\frac{x-1}{2} = \frac{y}{1} = \frac{z+1}{4}$ are
(a) parallel lines
(b) intersecting lines
(c) perpendicular skew lines
(d) None of these

17. The direction consines of the line drawn from $P(-5, 3, 1)$ to $Q(1, 5, -2)$ is
(a) (6, 2, –3) (b) (2, –4, 1)
(c) (–4, 8, –1) (d) $\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$

18. The coordinates of the centroid of triangle ABC where A, B, C are the points of intersection of the plane $6x + 3y - 2z = 18$ with the coordinate axes are
(a) (1, 2, –3) (b) (–1, 2, 3)
(c) (–1, –2, –3) (d) (1, –2, 3)

19. The intercepts made on the axes by the plane which bisects the line joining the points (1, 2, 3) and (–3, 4, 5) at right angles are

(a) $\left(-\frac{9}{2}, 9, 9\right)$ (b) $\left(\frac{9}{2}, 9, 9\right)$

(c) $\left(9, -\frac{9}{2}, 9\right)$ (d) $\left(9, \frac{9}{2}, 9\right)$

20. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is

(a) 4/3 (b) 2/3

(c) 3 (d) None of these

21. A variable plane passes through the fixed point (a, b, c) and meets the axes at A, B, C. The locus of the point of intersection of the planes through A, B, C and parallel to the coordinate planes is

(a) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$ (b) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$

(c) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=-2$ (d) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=-1$

22. A plane moves such that its distance from the origin is a constant p. If it intersects the coordinate axes at A, B, C then the locus of the centroid of the triangle ABC is

(a) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$

(b) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{p^2}$

(c) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{2}{p^2}$

(d) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{4}{p^2}$

23. The distance between two points P and Q is d and the length of their projections of PQ on the coordinate planes are d_1, d_2, d_3. Then $d_1^2 + d_2^2 + d_3^2 = Kd^2$ where K is

(a) 1 (b) 5

(c) 3 (d) 2

24. The line $\frac{x}{2}=-\frac{y}{3}=\frac{z}{1}$ is vertical. The direction cosines of the line of greatest slope in the plane $3x - 2y + z = 5$ are Proportional to

(a) (16, 11, –1) (b) (–11, 16, 1)

(c) (16, 11, 1) (d) (11, 16, –1)

25. The symmetric form of the equations of the line $x + y - z = 1, 2x - 3y + z = 2$ is

(a) $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$ (b) $\frac{x}{2}=\frac{y}{3}=\frac{z-1}{5}$

(c) $\frac{x-1}{2}=\frac{y}{3}=\frac{z}{5}$ (d) $\frac{x}{3}=\frac{y}{2}=\frac{z}{5}$

26. The equation of the plane which passes through the x-axis and perpendicular to the line $\frac{(x-1)}{\cos\theta}=\frac{(y+2)}{\sin\theta}=\frac{(z-3)}{0}$ is

(a) $x\tan\theta + y\sec\theta = 0$

(b) $x\sec\theta + y\tan\theta = 0$

(c) $x\cos\theta + y\sin\theta = 0$

(d) $x\sin\theta - y\cos\theta = 0$

27. The edge of a cube is of length of a. The shortest distance between the diagonal of a cube and an edge skew to it is

(a) $a\sqrt{2}$ (b) a

(c) $\sqrt{2}/a$ (d) $a/\sqrt{2}$

28. The equation of the plane passing through the intersection of the planes $2x - 5y + z = 3$ and $x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$ is $x + 3y + 6z = k$, where k is

(a) 5 (b) 3

(c) 7 (d) 2

29. The lines which intersect the skew lines $y = mx, z = c; y = -mx, z = -c$ and the x-axis lie on the surface

(a) $cz = mxy$ (b) $cy = mxz$

(c) $xy = cmz$ (d) None of these

30. The equation of the line passing through the point (1, 1, –1) and perpendicular to the plane $x - 2y - 3z = 7$ is

(a) $\frac{x-1}{-1}=\frac{y-1}{2}=\frac{z+1}{3}$.

(b) $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z+1}{3}$

(c) $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z+1}{-3}$

(d) none of these

31. The plane $4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $5x + 3y + 10z = 25$. The equation of the plane in its new position is $x - 4y + 6z = k$, where k is

(a) 106 (b) –89

(c) 73 (d) 37

32. A plane meets the coordinate axes in A, B, C such that the centroid of the triangle ABC is the point (a, a, a). Then the equation of the plane is $x + y + z = p$ where p is

(a) a (b) $3/a$
(c) $a/3$ (d) $3a$

33. If from the point $P(a, b, c)$ perpendiculars PL, PM be drawn to YOZ and ZOX planes, then the equation of the plane OLM is

(a) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ (b) $\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=0$
(c) $\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$ (d) $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0$

34. A variable plane makes with the coordinate planes, a tetrahedron of constant volume 64 k^3. Then the locus of the centroid of tetrahedron is the surface

(a) $xyz = 6k^2$
(b) $xy + yz + zx = 6k^2$
(c) $x^2 + y^2 + z^2 = 8k^2$
(d) none of these

35. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$, meets the co-ordinate axes at A, B, C such that the centroid of the triangle ABC is the point (a, b, c). Then k is

(a) 3 (b) 2
(c) 1 (d) 5

36. The perpendicular distance of the origin from the plane which makes intercepts 12, 3 and 4 on x, y, z axes respectively, is

(a) 13 (b) 11
(c) 17 (d) none of these

37. A plane meets the coordinate axes at A, B, C and the foot of the perpendicular from the origin O to the plane is P, $OA = a$, $OB = b$, $OC = c$. If P is the centroid of the triangle ABC, then

(a) $a + b + c = 0$ (b) $|a| = |b| = |c|$
(c) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ (d) none of these

38. A, B, C, D is a tetrahedron. A_1, B_1, C_1, D_1 are respectively the centroids of the triangles BCD, ACD, ABD and ABC; AA_1, BB_1, CC_1, DD_1 divide one another in the ratio

(a) 1 : 1 (b) 2 : 1
(c) 3 : 1 (d) 1 : 3

39. A plane makes intercepts OA, OB, OC whose measurements are a, b, c on the axes OX, OY, OZ. The area of the triangle ABC is

(a) $\frac{1}{2}(ab + bc + ca)$
(b) $\frac{1}{2}(a^2b^2 + b^2c^2 + c^2a^2)^{1/2}$
(c) $\frac{1}{2}abc(a + b + c)$
(d) $\frac{1}{2}(a + b + c)^2$

40. The projections of a line on the axes are 9, 12 and 8. The length of the line is

(a) 7 (b) 17
(c) 21 (d) 25

41. If P, Q, R, S are the points (4, 5, 3), (6, 3, 4), (2, 4, –1), (0, 5, 1), the length of projection of RS on PQ is

(a) $\frac{4}{3}$ (b) $\frac{2}{3}$
(c) 4 (d) 6

42. The distance of the point $P(-2, 3, 1)$ from the line QR, through $Q(-3, 6, 2)$ which makes equal angles with the axes is

(a) 3 (b) 8
(c) $\sqrt{2}$ (d) $2\sqrt{2}$

43. The direction ratios of the bisector of the angle between the lines whose direction cosines are l_1, m_1, n_1; l_2, m_2, n_2 are

(a) $l_1 + l_2, m_1 + m_2, n_1 + n_2$
(b) $l_1m_2 - l_2m_1, m_1n_2 - m_2n_1, n_1l_2 - n_2l_1$
(c) $l_1m_2 + l_2m_1, m_1n_2 + m_2n_1, n_1l_2 + n_2l_1$
(d) none of these

44. The points (8, –5, 6), (11, 1, 8), (9, 4, 2) and (6, –2, 0) are the vertices of a

(a) rhombus
(b) square
(c) rectangle
(d) parallelogram

45. The straight lines whose direction cosines are given by $al + bm + cn = 0$, $fmn + gnl + hlm = 0$ are perpendicular if

(a) $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$

(b) $\frac{a^2}{f}+\frac{b^2}{g}+\frac{c^2}{h}=0$

(c) $a^2(g+h)+b^2(h+f)+c^2(f+g)=0$

(d) none of these

46. The three planes $4y+6z=5$; $2x+3y+5z=5$; $6x+5y+9z=10$

(a) meet in a point

(b) have a line in common

(c) form a triangular prism

(d) none of these

47. The line $\frac{x+1}{2}=\frac{y+1}{3}=\frac{z+1}{4}$ meets the plane $x+2y+3z=14$, in the point

(a) $(3,-2,5)$ (b) $(3,2,-5)$

(c) $(2,0,4)$ (d) $(1,2,3)$

48. The foot of the perpendicular from $P(1,0,2)$ to the line $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ is the point

(a) $(1,2,-3)$ (b) $\left(\frac{1}{2},1,-\frac{3}{2}\right)$

(c) $(2,4,-6)$ (d) $(2,3,6)$

49. The length of the perpendicular from $(1,0,2)$ on the line $\frac{x+1}{3}=\frac{y-2}{-2}=\frac{z+1}{-1}$ is

(a) $\frac{3\sqrt{6}}{2}$ (b) $\frac{6\sqrt{3}}{5}$

(c) $3\sqrt{2}$ (d) $2\sqrt{3}$

50. The plane containing the two lines $\frac{x-3}{1}=\frac{y-2}{4}=\frac{z-1}{5}$ and $\frac{x-2}{1}=\frac{y+3}{-4}=\frac{z+1}{5}$ is $11x+my+nz=28$ where

(a) $m=-1, n=3$

(b) $m=1, n=-3$

(c) $m=-1, n=-3$

(d) $m=1, n=3$

Practice Test

MM : 20 Time : 30 Min.

(A) There are 10 parts in this question. Each part has one or more than one correct answer(s).

[$10\times 2=20$]

1. The projection of the line $\frac{x+1}{-1}=\frac{y}{2}=\frac{z-1}{3}$ on the plane $x-2y+z=6$ is the line of intersection of this plane with the plane

(a) $2x+y+z=0$

(b) $3x+y-z=2$

(c) $2x-3y+8z=3$

(d) none of these

2. A variable plane passes through a fixed point $(1,-2,3)$ and meets the co-ordinate axes in A, B, C. The locus of the point of intersection of the planes through A, B, C parallel to the co-ordinate planes is the surface

(a) $xy-\frac{1}{2}yz+\frac{1}{3}zx=6$

(b) $yz-2zx+3xy=xyz$

(c) $xy-2yz+3zx=3xyz$

(d) none of these

3. The distance of the point $(2,1,-2)$ from the line $\frac{x-1}{2}=\frac{y+1}{1}=\frac{z-3}{-3}$ measured parallel to the plane $x+2y+z=4$ is

(a) $\sqrt{10}$ (b) $\sqrt{20}$

(c) $\sqrt{5}$ (d) $\sqrt{30}$

4. The shortest distance between the lines $\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$ and $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$ is

(a) $2\sqrt{3}$ (b) $4\sqrt{3}$

(c) $3\sqrt{6}$ (d) $5\sqrt{6}$

5. The area of the triangle whose vertices are at the points $(2,1,1)$, $(3,1,2)$, $(-4,0,1)$ is

(a) $\sqrt{19}$ (b) $\frac{1}{2}\sqrt{19}$

(c) $\frac{1}{2}\sqrt{38}$ (d) $\frac{1}{2}\sqrt{57}$

6. The equation to the plane through the points (2, –1, 0), (3, –4, 5) parallel to a line with direction cosines proportional to 2, 3, 4 is $9x - 2y - 3z = k$ where k is
(a) 20 (b) –20
(c) 10 (d) –10

7. Through a point $P(f, g, h)$ a plane is drawn at right angles to OP, to meet the axes in A, B, C. If $OP = r$, the centroid of the triangle ABC is
(a) $\left(\frac{f}{3r}, \frac{g}{3r}, \frac{h}{3r}\right)$
(b) $\left(\frac{r^2}{3f^2}, \frac{r^2}{3g^2}, \frac{r^2}{3h^2}\right)$
(c) $\left(\frac{r^2}{3f}, \frac{r^2}{3g}, \frac{r^2}{3h}\right)$
(d) none of these

8. The plane $lx + my = 0$ is rotated about its line of intersection with the xOy plane through an angle α. Then the equation of the plane is $lx + my + nz = 0$ where n is
(a) $\pm\sqrt{l^2 + m^2}\cos\alpha$
(b) $\pm\sqrt{l^2 + m^2}\sin\alpha$
(c) $\pm\sqrt{l^2 + m^2}\tan\alpha$
(d) none of these

9. If a straight line makes an angle of 60° with each of the X and Y axes, the angle which it makes with the Z axis is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

10. The condition for the lines $x = az + b$, $y = cz + d$ and $x = a_1 z + b_1$, $y = c_1 z + d_1$ to be perpendicular is
(a) $ac_1 + a_1 c = 1$
(b) $aa_1 + cc_1 + 1 = 0$
(c) $bc_1 + b_1 c + 1 = 0$
(d) none of these

Record Your Score

	Max. Marks
1. First attempt	
2. Second attempt	
3. Third attempt	must be 100%

Answers

Multiple Choice

1. (c)	2. (c)	3. (c)	4. (b)	5. (d)	6. (c)
7. (a)	8. (d)	9. (b)	10. (b)	11. (a)	12. (c)
13. (d)	14. (d)	15. (b)	16. (c)	17. (d)	18. (a)
19. (a)	20. (c)	21. (b)	22. (b)	23. (d)	24. (d)
25. (c)	26. (c)	27. (d)	28. (c)	29. (b)	30. (c)
31. (a)	32. (d)	33. (c)	34. (a)	35. (a)	36. (d)
37. (b)	38. (c)	39. (b)	40. (b)	41. (a)	42. (d)
43. (a),(c)	44. (b)	45. (a)	46. (b)	47. (d)	48. (b)
49. (a)	50. (c)				

Practice Test

1. (a)	2. (b)	3. ((d)	4. (b)	5. (c)	6. (a)
7. (c)	8. (c)	9. (b),(d)	10. (b)		